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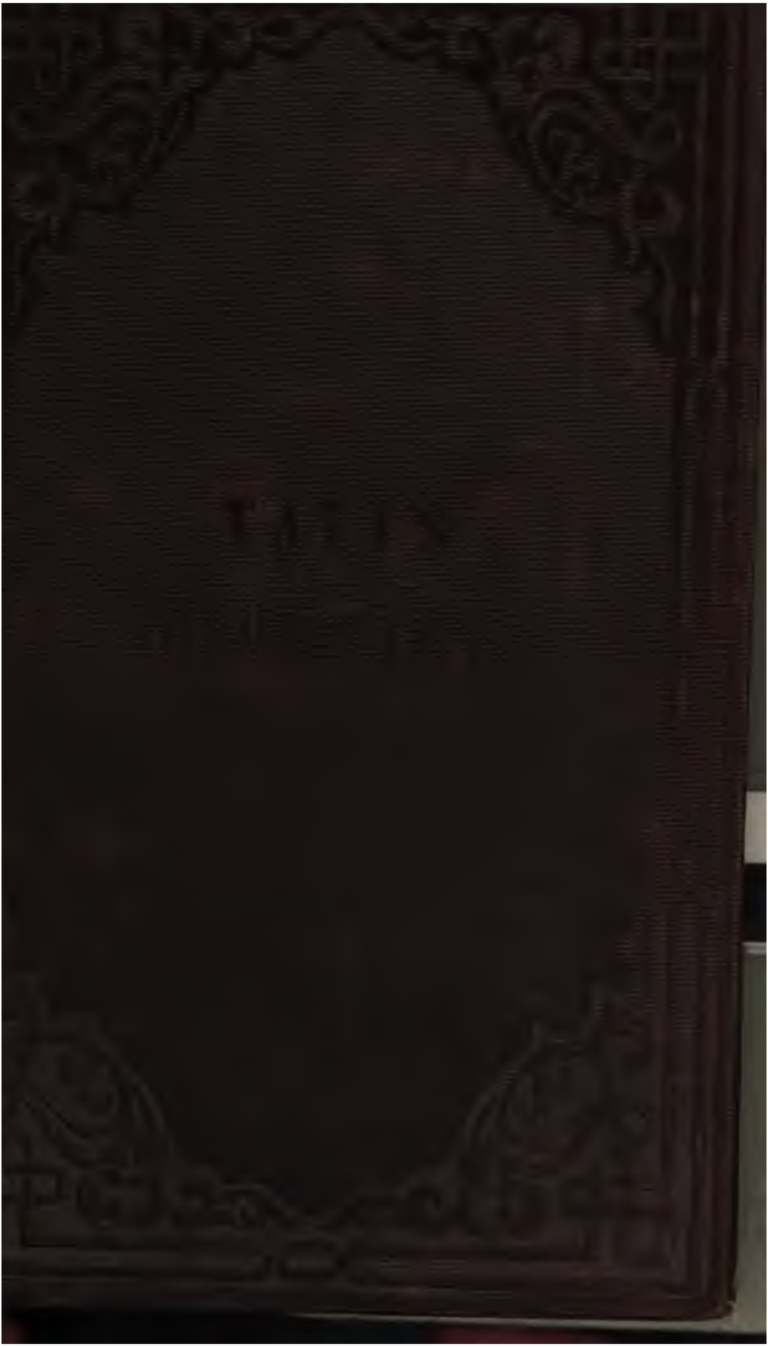

48. 1722.

PRINCIPLES

OF

GEOMETRY, MENSURATION,
TRIGONOMETRY, LAND-SURVEYING,
AND LEVELLING:

CONTAINING
FAMILIAR DEMONSTRATIONS AND ILLUSTRATIONS
OF
THE MOST IMPORTANT PROPOSITIONS IN EUCLID'S ELEMENTS;
PROOFS OF ALL THE USEFUL RULES AND FORMULÆ
IN
Mensuration and Trigonometry,
WITH THEIR APPLICATION TO THE SOLUTION OF
PRACTICAL PROBLEMS IN ESTIMATION, SURVEYING,
AND RAILWAY ENGINEERING.

BIBLIOTHECA · BODLEIANA

DESIGNED FOR THE USE OF SCHOOLS AND PRIVATE STUDENTS.

BY

THOMAS TATE,
MATHEMATICAL MASTER OF THE NATIONAL SOCIETY'S TRAINING COLLEGE,
BATTERSEA;
AUTHOR OF "EXERCISES IN MECHANICS, ETC.;" A TREATISE ON
"FACTORIAL ANALYSIS," ETC.

LONDON:
LONGMAN, BROWN, GREEN, AND LONGMANS,
PATERNOSTER-ROW.
1848.

London:
Spottiswoode and Shaw,
New-street-Square.

TO

J. P. KAY SHUTTLEWORTH, ESQ.

AND

E. C. TUFNEL, ESQ.

FOUNDERS OF THE BATTERSEA TRAINING COLLEGE,

This Work

IS

MOST RESPECTFULLY INSCRIBED,

AS

A TRIBUTE OF ESTEEM,

FOR THE IMPROVEMENTS THEY HAVE EFFECTED IN

OUR METHODS OF EDUCATION,

BY

THEIR HUMBLE AND MOST OBEDIENT SERVANT,

THE AUTHOR.

PREFACE.

THE present work is intended to supply Teachers and Practical Men with a simple and concise, yet comprehensive, system of theoretical and practical mathematics; embracing Geometry, Trigonometry, Mensuration, Surveying, and Levelling.

In a course of elementary instruction, Geometry is, perhaps, not less important than Algebra. It will, therefore, be instructive to trace the origin of our ideas in geometry, with the view of suggesting to us the means whereby first notions, on this subject, should be conveyed to the mind of the learner. Geometry, no doubt, claimed attention at a very early epoch in the history of the human race. Surrounded by objects of geometrical symmetry,—with all that is beautiful in form, in the earth, the sea, and the heavens,—man could not fail, even in the dawn of his intelligence, to trace the relations and properties of lines, surfaces, and solids. In the first rude essays at investigation, the scale and compasses would be used to confirm the deductions of reason, or the fortunate conjectures of intelligence; and a vast amount of geometrical facts would be accumulated, independently of the formality of definitions, or the tedious verbiage of a rigorous demonstration. The manner in which a figure is described, would invariably be taken as its definition; and thus no misunderstanding would take place with respect to the particular form considered, inasmuch as our conceptions arising from the perception of things, are always more vivid and positive, than those which are formed by mere verbal description. Results would be *esteemed more* valuable, than the processes by which they might be established. Without any precise views relative to the origin of ideas, or the formulæ of a technical logic, demonstration would consist in a simple appeal to common sense, or in such an exposition as might be sufficient to carry conviction to the mind. Practical applications and theoretical deductions, would act and react upon each other; new theorems would be followed by appropriate applications; and every new exigence in the business of life would open up fresh channels for theoretical research. Now the slow and inductive process by which truth is at first acquired, almost invariably gives us the rude features of the method, by which primary ideas, on the subject, should be communicated to the mind of the student. Hence, in the demonstrations contained in the following treatise, conciseness and simplicity have been preferred to the artificial verbiage of a technical logic; and those methods have been adopted, which seemed most in keeping with our ordinary associations, or which appeared to possess the most graphic and impressive features. Useful theorems have been selected, rather than those which are merely curious, or which have only a remote bearing upon the wants of the practical man. Demonstrations are followed by practical applications, calculated to elucidate the theorems, or to show their importance, and thereby to give the student confidence in his future efforts; and in some cases certain easy steps are omitted, with the view of exercising his inventive powers.

Elementary publications, generally, must necessarily change with the condition, tastes, and peculiar pursuits of society. To Euclid's Elements alone, of all other human productions, belongs the glory of having survived the dismemberment of empires, the changes of opinion, and the revolution in the tastes and habits of society: consecrated by time, and ennobled by its own intrinsic greatness, after the lapse of two thousand years, it still retains the suffrages of the literati of the world. Amid the mists and clouds of the false philosophy, and the enfeebling superstition of the dark ages, it shed a light over the nations; and at the dawn of modern philosophy, it appeared bright and clear, as the morning star of an age unparalleled in the history of humanity. And it may not be doubted, notwithstanding some logical defects, that long after the great mass of the writings of our own times shall have been consigned to oblivion, it will continue to form the text book of the learned. However, it must be conceded, that whatever may be its excellencies as a book of reference to the mathematician, its defects, as an initiatory system of geometry, are too apparent to admit of even an apology. A great book is, in many respects, a great evil; the very elements constituting its greatness,—its refinement and comprehensiveness,—tend to throw over it an air of mystery and dignity, which distracts and overawes the uninitiated student, in the place of giving him that encouragement and sympathy, which he certainly requires, in his first feeble efforts in the pursuit of abstract knowledge. The geometry of Euclid is a highly artificial system, which can only be read, thoroughly, by a person who is already a mathematician, and who can enter into its metaphysical subtilties, and beautiful yet operose demonstrations. The principle of motion gives a simplicity and clearness to many geometrical conceptions, but from an imagined inconsistency in the use of such a method, Euclid employs it, neither for the purpose of demonstration nor illustration. The method of superposition, which, in reality, lies at the very basis of geometrical demonstration, and, in many cases, gives a graphic interest to an investigation, is employed in the fourth proposition of his first book, and then, as if ashamed of the lowly origin of geometry, he scarcely uses it afterwards. Many of his problems are solved by methods which are never used in practice, for example, when a given portion is to be cut off from a straight line, instead of supposing the given portion to be simply transferred to, or placed *upon the* straight line, &c., which we really do in *practice, Euclid* must describe circle after circle, in order t accomplish the problem. The doctrine of similar triangles is, unquestionably, one of the most important propositions in the whole range of geometry, yet the student is not permitted to understand this proposition, until he has gone through the fifth book, which, to a large class of students, must for ever remain a sealed book. It is desirable that practical men should comprehend the leading propositions in solid geometry; but Euclid's method of treating this subject, is so operose and refined, as to place it beyond the reach of persons whose time for study is limited, or whose mathematical talents are not of a superior order.

The course of Analytical Trigonometry which I have given, is as simple and concise, as the nature and importance of the subject appear to admit of; and various practical applications of the theorems are given in connection with the demonstrations.

Many of the most essential rules of Mensuration are demonstrated in the geometrical course. The problems in Mensuration, have been selected solely on the ground of their utility. The Principles of Estimation, form a most important branch of Mensuration; indeed some really useful and intellectual treatise on this subject, has been long desiderated by practical men. The articles on Surveying and Levelling, will be found sufficiently comprehensive for the use of students of engineering.

Various original demonstrations are interspersed throughout the book. The generalization of Thomas Simpson's celebrated theorem, and the formulæ, relative to the most complex form of railway cuttings, are, I believe, essentially new.

That the work may prove useful to the class of persons, for whose instruction it is specially designed, is the earnest wish of the Author.

T. TATE.

March, 1848.

DIRECTIONS FOR THE STUDENT.

Before commencing the study of geometry, it is desirable that he should be able to solve a simple equation of ordinary difficulty. When he has studied as far as page 66, he may solve some of the most useful problems in Mensuration, and then read consecutively, the application of Algebra to Geometry, Trigonometry, Mensuration, Estimation, Surveying, and Levelling, omitting those parts which involve a knowledge of the more difficult theorems in Solid Geometry. The foot notes may also be omitted, in reading the book for the first time. See also the directions given at page 83.

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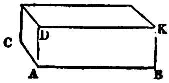
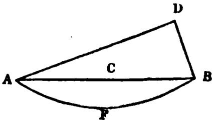
PRINCIPLES OF GEOMETRY.

PRELIMINARY IDEAS ON GEOMETRY. DEFINITIONS, AND FIRST PRINCIPLES.

1. GEOMETRY is that branch of science which teaches the properties of lines, surfaces, and solids. Every body has three dimensions, length, breadth, and thickness or depth. Thus in a block of wood, AB is the length, AC is the breadth, and AD is the thickness or depth. The boundaries or limits of bodies are faces or *surfaces;* thus ADKB is a surface: thus we speak of the surface of the table without any regard to its thickness, or of the surface of a lake without any regard to its depth. The boundaries or limits of surfaces are *lines;* thus the edge AB is a line: thus we speak of the edge or line limiting the surface of the table without any regard to its extent. When lines intersect or cross each other, the *place* where they intersect is called a *point.*

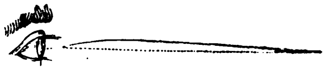




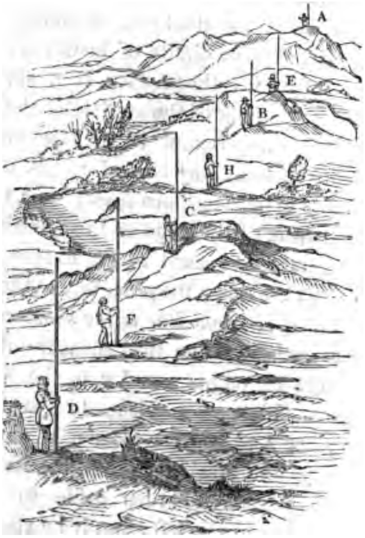
2. *A straight line is the shortest path or distance between two points.* The true measure of the distance between two places or points A and B is the length of the straight line ACB. If a person wishes to travel by the shortest path from a place A to another place B, he moves along the straight line ACB, taking care neither to turn aside to the right nor to the left; but if he should move along the crooked line ADB or the *curved* line AFB, he will have gone over a greater

distance than if he had gone in the direct line ACB. Hence it follows that in the *triangle* ADB, *the two sides* AD *and* DB *put together are greater or longer than the third side* AB.

3. To make a ***straight edge*** or ruler, the carpenter brings his eye on a level with the edge, and if he sees every part of the line from end to end, he is satisfied that the edge is straight; but if he only gets a sight of the middle at the same time that he sees one extremity, he concludes that the ruler is round; and on the contrary, if he sees both extremities without seeing the middle, he concludes that the ruler is hollow. This process depends upon the fact that the rays of light proceed from objects to the eye in straight lines.

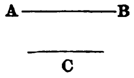


4. On the same principle, if a surveyor wants to lay down a straight line upon the ground between the points A and D, he puts down staffs or piquets at different intermediate points; then, by bringing his eye in a line with the extreme staffs A and D, he is enabled to see when the intermediate staffs are in the straight line joining A and D. Or if he requires to continue or produce a straight line joining any two staffs, say A and E, to the station B, he brings the staff B to a certain position where he sees that the three staffs, B, E, and A *cover* each other, then this point B will be in the straight



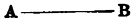
line A E produced; and so on to any other point. In this way a straight line may be traced for miles through a country over hills and valleys, with any degree of precision. After the same method a surveyor measures the length of any straight line in a field by means of the chain and arrows, which serve the place of staffs.

5. If a straight line C be moved along another straight line A B, the lines will exactly fit each other in every position; for otherwise the lines would not be the shortest path between the points where they did not correspond. If two persons go in a straight line between two places, they will evidently pursue the same path.



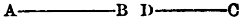
6. The following questions are intended to show how the teacher may communicate much valuable information in geometry by a suggestive method of interrogation. Nearly all the geometrical knowledge contained in this work may be conveyed to the pupil in this manner.

Teacher. What is the line A B called?



Pupil. It is called a straight line.

T. Of the two straight lines A B and D C, which is the greater?



P. The line A B is the greater.

T. How should you ascertain this with certainty?

P. By laying the line D C upon A B.

T. What sort of line is A F B (*see fig.* Art. **2.**)?

P. It is a crooked line.

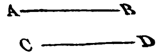
T. True; but it is also called a *curved* line. Whether is the curved line A F B or the straight line A B the shorter?

P. The straight line A B.

T. If you wanted to go from Battersea school to the church, in what line should you walk?

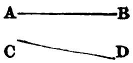
P. In a straight line. (Why?) Because a straight line is the shortest distance between the school and the church.

T. What have you to say relative to the two *straight lines* A B and C D?



P. They appear to be of the same length; and moreover they appear to lie even with each other.

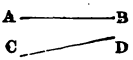
T. In other words you might say, C D = A B; and also C D is *parallel* to A B. Is C D now parallel to A B?



P. No; for C D would meet A B on the left side.

T. On which side would they now meet?

P. On the right hand side.

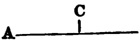


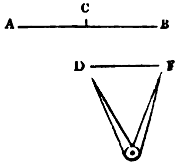
T. What is therefore the peculiar property or definition of parallel lines?

P. That if they be carried out ever so far, on either side, they will never meet.

7. A surface is called a *plane*, or flat even surface, when the line between any two points upon it is straight. Thus the surface of the table is a plane if a straight-edge exactly fits it when applied in every direction. To ascertain when a surface is a plane, bring your eye on a level with it, and if you find that every point in the surface can be seen at the same time, it will show that the surface is a plane. Our figures are supposed to be drawn on planes.

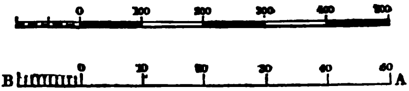
8. *To measure lines.* — When a workman wants to ascertain whether or not two rods are of the same length, he lays the one rod upon the other, taking care to bring two of the ends together; then if the other extremities correspond, he concludes that the rods are of the same length. This process is, in fact, one of the first steps in demonstrative geometry. Let A———B and C———D be two right lines of the same length; if C D be laid upon A B, so that the extremity D of the one shall correspond to the extremity B of the other, then the extremity C of the one will exactly correspond with the extremity A of the other. If D C be longer than B A, the extremity C will overlap A; and on the contrary, if D C be shorter than B A, the extremity C will not *reach to the extremity* A.

If C be the middle of the line , and if C B be folded over on C, C B will exactly cover the line C A. The line A B is said to be bisected in the point C.

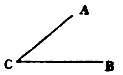


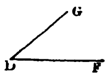
The compasses are usually employed for measuring off equal lines. Thus, on the line A B, if we want to mark off B C equal to F D; with the compasses we simply take off the line F D, and apply it to B C, and the thing is done.

To measure the length of a line it is necessary that we should fix upon some unit of measure, and then the number of times that this unit must be repeated to make up the given line, will be the units of length in that line. Thus if the measuring unit be an inch, the length of the line will be found in units of inches; if the measuring unit be a foot, the length of the line will be found in units of feet; and so on. If the measuring unit is not exactly contained in the proposed line, the length of the remaining portion must be expressed in certain fractional parts of the unit.

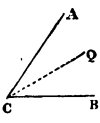


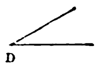
To construct a scale of equal parts for comparing the length of lines, take any small opening of the compasses and repeat it ten times, making up the part B 0; with an opening of the compasses equal to B 0 mark off the equal divisions 10, 20, 30, &c., then taking each division in B 0 as our unit, the distance from 0 to 10 will be 10 units, from 0 to 20 will be 20 units, and so on. If each division in B 0 is taken as 10 units, then the divisions on the scale equal to B 0 will be hundreds. If B 0 be taken as a unit, then each division in B 0 will be one-tenth; thus when B 0 represents a foot, A 0 will represent 5 feet; and each subdivision in B 0 will represent one-tenth of a foot.



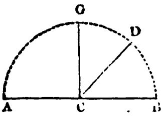


9. *Angles and their measure.* An angle is the opening formed by two straight lines meeting each other. Thus the lines AC and CB form the angle C, and then AC is said to be inclined to the line CB at the angle ACB or BCA, where the letter C at the angular point is put between the other two letters. The symbol ∠ is frequently used to denote an angle; thus in the place of writing angle ACB we may write ∠ACB. When no mistake can arise, we simply write angle C or ∠ C. Two angles are equal when they exactly fit each other. Thus in order to ascertain whether or not the ∠ D is equal to the ∠ C; we lay the lines forming the ∠ D upon the lines forming the ∠ C, in such a way that the point D shall fall upon C, and the line DF shall cover the line CB; then if DG covers CA the angles will be equal, but if DG should fall without CA, then the ∠ D will be greater than the ∠ C, and on the contrary if DG should fall within CA, then the ∠ D will be less than the ∠ C. It will be readily seen, that the angle does not at all depend upon the length of the lines which form it, for they may be extended or shortened without altering the angle.



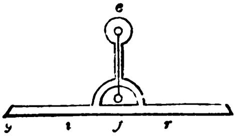


The ∠ ACB is double the ∠ D, when the ∠ D must be repeated two times to make up the ∠ ACB. If the line CQ divides the ∠ ACB into two equal parts, the line CQ is said to bisect the angle; and if ∠ QCB be folded over on CQ, the line CB will exactly fall upon the line CA.

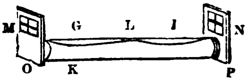


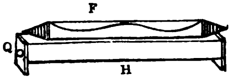
The line DC meeting the line AB evidently inclines more to the right than to the left, that is, the ∠ DCA is greater than the ∠ DCB: but if CG be drawn so that it shall neither incline to the right nor to the left, thereby making the ∠ GCB = the ∠ GCA, then these angles are called *right angles*, and the line GC is said to be *perpendicular to the* line AB. The angle DCB is called *acute*, because it is less than a right angle; and the angle DCA is called *obtuse*, because it is greater than a right angle.

10. *Illustrations and practical applications.*—If a line carrying a plummet or round piece of lead be held over smooth or level water, the line will be perpendicular to the surface of the water.

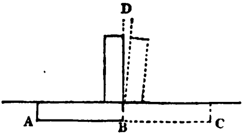


The plummet level is constructed on this principle. When the plummet line *ef* exactly falls upon a line drawn upon the wood perpendicular to *gr*, then the line *gr* is a level line.



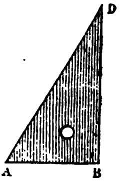


Spirit levels are also used by workmen for laying stones and timbers level. The instrument consists of a glass tube GI, nearly filled with spirits of wine, mounted in a stand or frame OKP. At the ends N and M there are two slits made at right angles to each other. To use the instrument, elevate or depress one of the extremities until the air bubble in the tube is exactly in the centre L; then the tube is level, and OP is a level line. By placing the eye at the centre of the slit N, and looking through the centre of the opposite slit, the level line may be extended to any distance. Sometimes the sight-hole Q is made below the spirit tube, as in the second figure here shown.

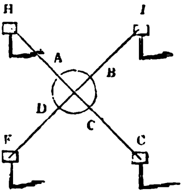


A square ABD is an instrument formed by joining two straight pieces of wood at right angles to each other. When a workman wants to adjust a square, he places the stock AB along the edge of a plank and draws the line BD along the blade of the square; he then reverses the instrument and draws another line along the blade; if these two lines exactly correspond, he concludes that the square is correct, but if they do not he is enabled to see what must be done to the edge of the blade in order to make the square perfect.

A triangular piece of wood ABD, having the edge BD at right angles to AB, is very convenient for drawing perpendicular lines on paper. There is a hole cut in the centre of the wood, to enable the draftsman to move the instrument with facility.

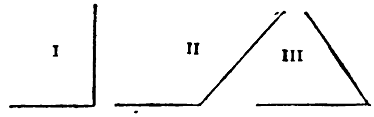


The cross staff is a simple instrument, used by surveyors, for laying down lines on the ground at right angles to each other. It consists of a circular piece of wood ABCD, about six inches in diameter, having two saw slits AC and DB cut at right angles to each other. This circular piece of wood is placed upon a staff which has a pointed end to enable the surveyor to push it into the ground. In order to show the use of the instrument, suppose the surveyor has to lay down a line HC perpendicular to the straight line KI, connecting the piquets K and I. 1st. To place the staff in the line KI; look along the slit BD, first towards the piquet K and then towards I, and move the instrument until it is found that the slit ranges with both piquets. 2nd. Look along the slit CA, and let an assistant put down the piquets H and C in the line of sight; then HC will be perpendicular to KI.



11. *Teacher.* Name the angles here drawn.

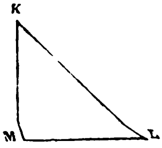
Pupil. No. I. is a right angle; No. II. is an obtuse angle; and No. III. is an acute angle.



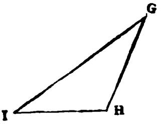
T. How many angles has the triangle MLK?

P. It has three angles; and therefore it is called a triangle.

T. But this particular form of a triangle *is called a right-angled* triangle. Why?

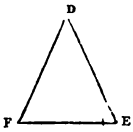


P. Because one of its angles M is a right angle.



T. The triangle IHG is called an obtuse-angled triangle. Why?

P. Because one of its angles H is an obtuse angle.



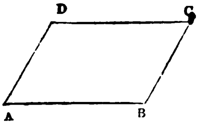
T. This triangle is an acute-angled triangle. When do triangles receive this name?

P. When all the angles are acute.

T. In the last figure what sides appear to be equal?

P. All the sides appear to be equal to each other.

T. When this is the case the triangle is called an *equilateral* or equal-sided triangle; and when only two sides are equal the triangle is called *isosceles.*



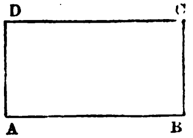
T. In this *parallelogram* which sides are drawn parallel?

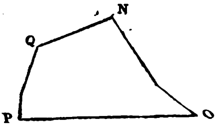
P. The sides opposite to each other are parallel, that is, AB is parallel to DC, and AD to BC.

T. What you have just said is the definition of a parallelogram. What is the form of your slate?

P. It has the shape of a parallelogram.

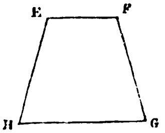
T. True; but because one side is perpendicular to the other, it is also called a *rectangle.* Thus ABCD is a rectangle, where the opposite sides are not only parallel to each other but the ∠ B, included by the adjacent sides, is a right angle. When the length of a rectangle is the same as its breadth, that is, when AB = BC, then the figure is called a *square.*





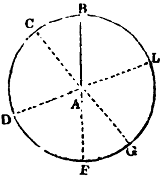
T. The four-sided figure PONQ is called a *trapezium.* What have you to say relative to the direction of the sides?

P. None of the sides are parallel to each other.



T. This figure is called a *trapezoid.* Is there any thing which distinguishes this figure from the preceding one?

P. The side EF is parallel to HG.



T. (Drawing a circle with a string.) Having seen this *circle* described, tell me how the *circumference* BLFDC is situated with respect to the *centre* A?

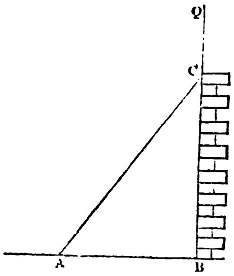
P. Every point in the circumference must be at the same distance from the centre, that is, AB, AL, AG, AF, &c. must be all equal to each other.

T. Any line AB drawn from the centre to the circumference is called the *radius,* and a line BAF drawn through the centre is called the *diameter.* Now what is the length of the diameter as compared with the radius?

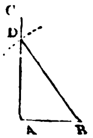
P. The diameter will be double the radius; because the diameter BF is made up of the radii AB and AF.

Problems.

1. The height of a wall BC is 20 ft., required the length of a ladder AC which will reach to the top, when the foot of the ladder is 15 ft. from the wall?

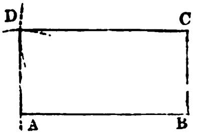


From a scale of equal parts, Art. 8., take off AB = 15, and with the triangular square draw BQ perpendicular to AB; from the same scale of equal parts take BC = 20; join AC, and measure this distance by the scale, and it will give the length of the ladder required = 25 ft.



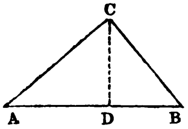
2. There are two roads AB and AC at right angles to each other, and at B and D there are two houses; what distance would there be saved *in going from* B to D in a direct line, supposing *AB = 150 yds.,* and AD = 180 yds.? *Ans.* 96 yds.

3. Draw the ground plan of a house 37 ft. long, and 20 ft. broad; and find the length of the diagonal line drawn from A to C.

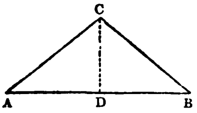


From a scale of equal parts take off AB = 37; with the triangular square draw BC and AD perpendicular to AB; from the same scale take BC and AD each equal to 20; join CD, and ABCD will be the plan required. Extend the compasses from A to C and apply it to the scale, and it will give the length of the diagonal required = 42 ft. nearly.

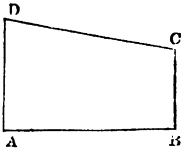
4. In surveying a triangular field ABC with the chain and cross staff, I measured along the side AB until I came to the point D, where a line DC drawn to the corner C is perpendicular to AB; I then found AD = 7 chains, DC = 6 chains, and DB = 5 chains. Give a plan of the field, and find the lengths of the sides AC and CB. *Ans.* AC = 9·2 chains, and BC = 7·8 chains.



5. Let ABC be the drawing of a roof, where the perpendicular height CD is 8 ft., and the half breadth AD of the tie-beam is 8·5 ft.; required the length of the rafter AC. *Ans.* 11·6 ft.



6. Wishing to know the distance between two objects D and C inaccessible from each other, I measured a line DA = 6 yds., I then went in a straight line AB at right angles to AD; coming opposite to the object C, I put down a staff at B, making BC at right angles to BA; I then found AB = 9 yds., and BC = 4 yds. Required the distance between D and C.



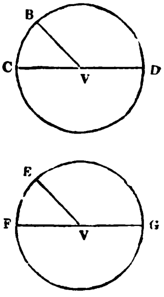
From a scale take AB = 9; erect the perpendiculars AD and BC; *from the* same scale take AD = 6, and BC = 4; join

DC and measure it by the scale, and it will be the distance required $\doteq$ 9·2 ft.

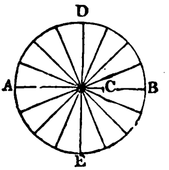
7. After walking in a direct line the distance of 30 yds., I turned in a perpendicular direction to the left; after going 60 yds. in this line, I turned in a perpendicular direction to the right; and after going 50 yds. in this line I returned by a straight path to the place from which I set out. What distance did I walk over? *Ans.* 240 *yds.*

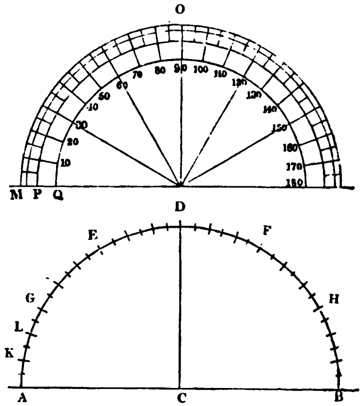
12. *The diameter divides the circle into two equal parts.* For if BCDF be folded over upon BF, the curved line BCDF will exactly cover the curved line BLGF, because every point in these curved lines is at the same distance from the centre A. Hence BCDF is called a *semicircle.* (See *fig.* p. 10.)

13. *Circles that have equal radii are equal, and equal arcs will form equal angles at the centre.* For let the circle FEG be laid upon the circle CBD, so that the centre V of the one shall fall upon the centre V of the other, and the diameter FG of the one upon CD of the other, then, as the circles have the same radii, the circumferences will exactly coincide; and the equal arcs FE and CB will exactly coincide; therefore the line EV will fall upon the line BV, and thus $\angle$ FVE = $\angle$ CVB. Straight lines joining the points F and E, C and B will also coincide. The space CVB is called a *sector* of the circle.



If the circumference of the circle ADBE be divided into any number of equal parts, the consecutive lines joining the points of division and the centre of the circle will be inclined to each other at the same angles; for by what has just been explained all these sectors will exactly fit each other. Hence it follows, that equal arcs contain equal angles, *and conversely* equal angles contain equal arcs.

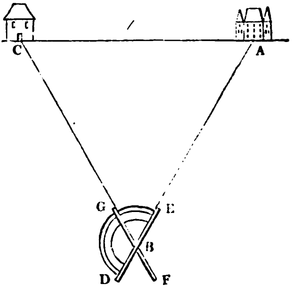




If the circumference of the circle be divided into 360 equal parts, each part is called a degree, and the angle formed by any two consecutive lines drawn from the points of division to the centre D of the circle will be an angle of 1 degree, or 1°. Taking 30 of these divisions, for example, the ∠ QD 30 will be an angle of 30 degrees or 30°; it will be readily seen that the angle thus formed will be 30 times the angle formed by drawing a line to the first division on the arc. The line DO being drawn perpendicular to MD, the angle MDO will be a right angle, and the arc MO will be a quadrant or a quarter of the circle, and therefore will contain $\frac{1}{4}$ of 360°, or 90°. The semicircle will contain $\frac{1}{2}$ of 360°, or 180°, which is also equal to 2 right angles. If a degree be divided into 60 equal parts, each part is called a minute; and if the minute be divided into 60 equal parts, each part is called a second. We write, for example, 36 degrees 14 minutes 25 seconds in the form 50° 14′ 25″. The instrument here described is called *a protractor;* it is usually made of brass, and is very useful for laying down lines having a given inclination to each other. For the purpose of school instruction it may be made of card paper with a radius of about six inches. From the point C, let it be required to draw a line making an arc of 60° with the line AC. Place the centre D of the protractor on the given point C, and the edge DM along the line CA, then make a mark E corresponding to the 60° on the instrument, and join the points E and C, and ∠ ACE will contain 60°.

If the quadrant AD be divided into three equal parts AG, GE, and ED, each part will be an arc of $\frac{1}{3}$ of 90°, or 30°; and if AG be divided into three equal parts AK, KI, and IG, each part will be an arc of $\frac{1}{3}$ of 30°, or 10°, and so on to any other subdivisions.

14. It will be useful for the student to know, at this stage of his instruction, something about the way in which angles are taken by the surveyor. The theodolite is the most useful instrument for measuring the angle contained by lines drawn from a point to two distant objects. The principle upon which this instrument is constructed may be readily understood. Let DGE be a circle divided into degrees like the protractor; upon the centre B of this circle let a tube or spy-glass DE turn upon a pivot; direct the tube first towards the object A, and then turn it until it comes to the position FG, in a line with the other object C; then the angle formed by the lines AB and CB will be measured by the number of degrees in the arc EG.



Problems.

1. In order to find the distance of A from the station B, without actually measuring it, the surveyor finds the ∠ ABC (=40°); he then measures the distance BC (= 300 yds.), and placing his theodolite at C finds the ∠ BCA (= 70°).

Here, by construction, from the scale of equal parts, Art. 8., take off BC = 300; with the protractor* draw a line BA, making the ∠ ABC = 40°, and CA, making the ∠ ACB = 70°; then these lines will intersect each other in a point A; with the compasses take off BA, and apply it to the scale of equal parts, and the units on the scale will be the units of yards in the required distance = 205 yards.

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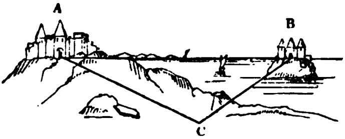
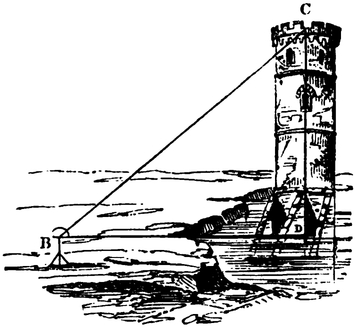
* *Or with* the scale of chords, hereafter described.

2. To find the distance of two towers A and B, inaccessible from each other. With the theodolite take the ∠ C (= 110°), measure A C (= 32 yds.), and with the theodolite at A take the angle A (= 30°), which the tower B makes with a staff placed at C; then we find by construction A B = 46 yards.



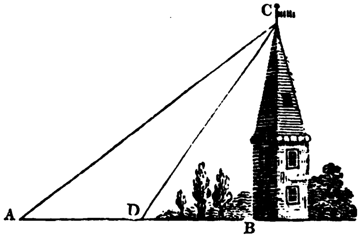
3. To find the height of a tower D C. The surveyor measures from the bottom D of the tower the horizontal line DB = 400 ft.; he then places his theodolite at B, and takes the angle CBD (= 40°), which the top of the tower makes with the level line BD. Required the height DC of the tower.



From the scale of equal parts take B D = 400; with the protractor draw BC, making the ∠ B = 40°, and with the square, Art. **10.**, draw DC perpendicular to BD; then these lines will intersect each other in a point C. With the compasses take off the height DC, and apply it to the scale of equal parts, and the units upon the scale will give the units of feet in the height of the tower = 335 ft. nearly.

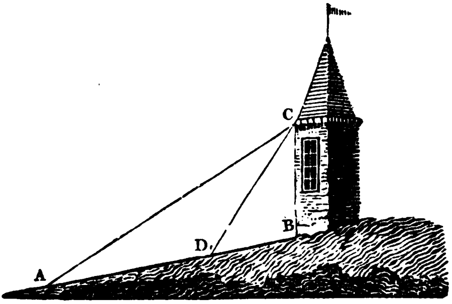
4. To find the height of a steeple BC, whose base B is inaccessible. The surveyor measured a base line AD = 76 ft.; he then took the ∠ C D B = 52°, and the ∠ CAB = 27° Required the height C B of the steeple.

From a scale of equal parts mark off AD = 76; with the protractor draw D C, making the ∠ D = 52°, and draw AC.



making the ∠ A = 27°, then the point C, where these two lines intersect each other, will be the top of the steeple; with the square draw CB perpendicular to AB; by the scale of equal parts measure the units in CB, and they will give the units of feet in the height, as required, = 64 ft.

5. To find the height of a tower CB standing upon a hill AB. The surveyor measured in the same straight line BD = 50 ft., and DA = 75 ft.; he then found the ∠ CDB = 41°, and ∠ CAB = 24°. Required the height CB of the tower.



Draw any line AB, and from a scale of equal parts take off BD = 50, and DA = 75; with the protractor draw DC, making the ∠ D = 41°, and AC, making the ∠ A = 24°; then these two lines will intersect each other in a point C; join C and B, and CB measured off the scale of equal parts will give the height required = 69 ft.

Explanation of Terms and Processes.

15. A *definition*, in geometry, explains the peculiar way in which a geometrical line or figure is formed. Thus, when we say that an isosceles triangle is a triangle that has *two of* *its sides equal to each other*, we *define* this sort of triangles. The definitions of figures form the premises,—the properties admitted,—upon which our conclusions, or properties to be demonstrated, are based. Thus from the definition of an isosceles triangle we can prove that the angles opposite to the equal sides are also equal. The property to be proved in any figure is called a theorem. Too much importance can scarcely be attached to the clearness of our definitions; and care should be taken to distinguish between the properties which we *assume* and those which we must *prove*. It matters not, so far as our conclusions are considered, whether or not the figures, which we define, be actually described, provided we conceive them to be so according to the form of the definition; nor is it necessary, as in the course of demonstration, that we actually place one figure upon another, provided we conceive the thing to be done in the manner described. The fundamental *method* of geometrical demonstration is that of *superposition*, or the placing of one figure upon another.

The *converse* form of a theorem consists in stating the terms of the theorem in a reverse order. Such theorems are usually proved by the method called *reductio ad absurdum*, or the reducing of all other suppositions to an absurdity. In most cases the converse form of a theorem is really involved in the very proof we give of the theorem itself.

An *axiom* is an intuitive or self-evident truth, that is, a truth which is admitted the moment the terms in which it is expressed are understood. If equals be added to equals, the wholes are equal; if equals be taken from equals, the remainders are equal; things which are equal to the same thing, are equal; &c., are examples of axiomatic truths. The student will have no difficulty in observing, in the course of any demonstration, what principles are taken as axioms. It must be conceded that some theorems are as simple, and are as readily admitted, as the axioms upon which they are made to depend. In this class may be ranked the properties of parallel lines. Such propositions, in a first course of geometry, may be taken as axioms without materially infringing upon the logical strictness of the geometrical course.

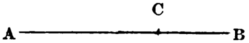
In a *problem* there is something required to be done from certain things that are given, which are called the data of the problem. Theorems, as well as problems, may be regarded as *propositions,* inasmuch as in both cases there is something *proposed* to be done.

In *postulates* it is assumed that it is *possible* to draw figures exactly as they are described in our definitions. Thus, although we may not actually draw a parallelogram, yet it is obvious that it is possible to construct this figure in accordance with its definition.

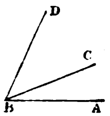
A *corollary* is an obvious deduction from one or more propositions.

An *hypothesis* is something which we suppose to be true, either in the enunciation of a theorem or in the course of its demonstration.

The sign of equality is $=$; thus $A = B$, signifies that A is equal to B. When two or more quantities are to be added, the sign $+$ or plus is used; thus $AB = AC + CB$, for here the whole line is made up of the two parts AC and CB. When one quantity is to be taken from another, the sign $-$ or minus, is used; thus $AB - CB = AC$, for when the part CB is taken away from AB, the part AC must be left.

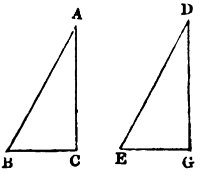


In the case of angles we have, $\angle ABD = \angle ABC + \angle CBD$, for the whole angle is made up of these two parts; and also $\angle ABD - \angle CBD = \angle ABC$, for when one of the parts is taken away from the whole angle, the other part must be left.



The symbol $\therefore$ signifies *therefore;* indeed, every symbol used in algebra admits of the same interpretation in geometry.

THEOREMS AND PROBLEMS RELATIVE TO ANGLES, TRIANGLES, AND PARALLEL LINES.

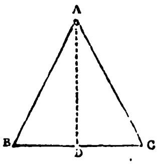


16. Theorem. The triangles BCA and EGD are equal in all respects, when BC = EG, BA = ED, and ∠ B = ∠ E.

For let BCA be laid upon EGD, so that the side BC shall lie on its equal side EG; then as ∠ B = ∠ E, the line BA will lie along ED; and because BA = ED, the extremity A will fall upon the extremity D, and therefore the triangle BCA will exactly fit or cover the triangle EGD.

17. Theorem. The triangles BCA and EGD are equal in all respects, when BC = EG, ∠ B = ∠ E, and ∠ C = ∠ G.

For let BCA be laid upon EGD, so that the side BC shall lie on its equal side EG; then as ∠ B = ∠ E, the line BA will lie along ED; and because ∠ C = ∠ G, the line CA will lie along GD, that is, the triangle BCA will exactly fit the triangle EGD.

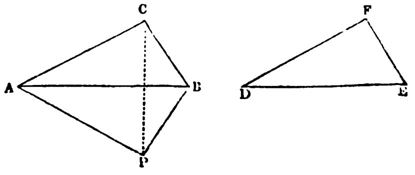


18. Theorem. In the isosceles triangle ABC, where AB = AC, the angles opposite the equal sides are equal, that is, ∠ B = ∠ C.

For let AD be a line dividing the ∠ A into two equal parts. Now let the triangle ADB be folded over on the line AD: then because ∠ DAB = ∠ DAC, the line AB will fall upon AC; and because AB = AC, the point B will fall upon C, and the triangle ADB will fit exactly to the triangle ADC, and therefore ∠ B = ∠ C.

Corollary. The line AD bisecting the vertical angle A, divides the base into two equal parts, that is, BD = DC; and moreover AD will be perpendicular to BC.

19. Theorem. The triangles DEF and ABC are equal in all respects, when the three sides of the one are respectively equal to the three sides of the other, that is, *DE = AB, DF = AC,* and EF = BC.



Let DEF be placed below ABC, so that DE shall coincide with AB, and the vertex F shall fall at P; join the points C and P; then the triangles ACP and BCP are isosceles (why?), and therefore by the preceding theorem,

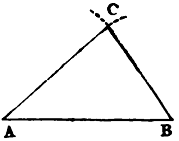
$$\angle ACP = \angle APC, \text{ and } \angle BCP = \angle BPC,$$

adding these equals together,

$$\angle ACP + \angle BCP = \angle APC + \angle BPC, \text{ or } \angle ACB = \angle APB.$$

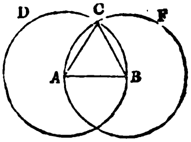
Hence it follows, Art. **16.**, that the triangles ABC and ABP, or DEF, are equal in all respects, for $AC = AP$, $BC = BP$, and $\angle ACB = \angle ABP$.

Exercise. It is required to construct a triangle, having given the three sides, *viz.*, $AB = 27$, $AC = 22$, and $BC = 18$.



From the scale of equal parts take off $AB = 27$; on A as a centre with the radius $AC = 22$ sweep a circle, and on B as a centre with the radius $BC = 18$ sweep a circle intersecting the former in the point C; join A and C, B and C, and ABC will be the triangle required.

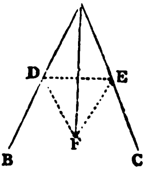
20. PROBLEM. On a given line AB to describe an equilateral triangle ABC.



On A as a centre with the radius AB describe the circle BCD; and on B as a centre, with the same radius BA, describe the circle ACF, intersecting the former in the point C; join A and C, B and C, and ACB will be the equilateral triangle required.

Here $AC = AB$, being radii of the circle BCD, and $BC = AB$, being radii of the circle ACF; therefore $AC = BC$, because each of them is equal to the same thing, *viz.* AB.

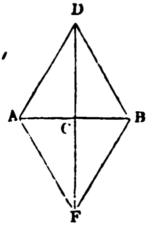
21. PROBLEM. To bisect a given $\angle BAC$.



Take any point D in the line AB, and cut off $AE = AD$; join D and E; on the side DE describe the equilateral triangle DEF (Art. **20.**), and join F and A. Then AF will bisect the $\angle BAC$.

The triangles ADF and AEF are identical; for $AD = AE$, $DF = EF$, and AF belongs to both triangles; therefore by Art. **19.** the triangles are in all respects equal, and $\angle FAD = \angle FAE$. If the triangle ADF be folded over on the line AF, it will exactly cover the triangle AEF.

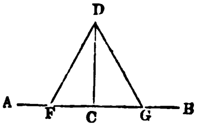
22. Problem. To bisect a given line AB.



Upon AB describe the equilateral triangles ABD and ABF (Art. **20.**); draw the line FD, cutting AB in the point C, then $AC = CB$.

Precisely as in the last problem, it may be shown that $\angle ADF = \angle BDF$; then in the triangles ADC and BDC, we have $AD = BD$; DC common to both, and $\angle ADC = \angle BDC$; therefore, by Art. **16.**, the triangles are identical, and $AC = CB$.

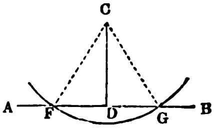
23. Problem. From a given point C in the straight line AB to draw the perpendicular CD.



With any opening of the compasses mark off $CF = CG$; on the side FG construct the equilateral triangle FGD (Art. **20.**); join the points D and C; and then CD will be the perpendicular required.

For in the triangles FDC and GDC, $FC = CG$, $FD = GD$, and DC belongs to both triangles; therefore by Art. **19.** the triangles are identical, and $\angle DCF = \angle DCG$; that is, DC is perpendicular to AB.

24. Problem. From a given point C to let fall the perpendicular CD upon the straight line AB.



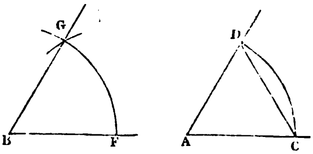
On C as a centre with a sufficient opening of the compasses describe a circle cutting AB in the points F and G; join F and C, G and C; by Art. **21.** draw CD bisecting the $\angle FCG$; then CD will be the perpendicular required.

If the triangle CDG be folded over on the line CD, it will exactly cover the triangle CDF, because $\angle DCG = \angle DCF$, and $CG = CF$. Therefore $\angle CDG = \angle CDF$, that is, CD is perpendicular to AB.

25. **Theorem.** The perpendicular DC is the shortest line that can be drawn from the point D to meet the straight line AB. See fig. to Art. **22**.

Let DC be produced until $CF = CD$, and take any point A in the line AB; join AF and AD; then $\angle ACD$ and $\angle ACF$ will be right angles. If the triangle ACF be folded over on the line AC, it will exactly cover the triangle ACD (why?), and therefore $AF = AD$. Now by Art. **2**., $AD + AF$ is greater than DF; or, what is the same thing, 2 AD is greater than 2 CD; and hence AD is greater than CD. The same thing may be shown for any other point in the line AB.

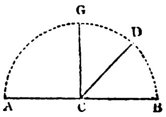
26. **Problem.** To make the $\angle B$ equal to the given $\angle A$.



On A as a centre with any opening of the compasses describe the arc CD, and on B as a centre, with the same opening of the compasses, describe the arc FG; with the compasses take off CD, and on F as a centre describe an arc cutting the former in the point G; join B and G, and then $\angle FBG = \angle CAD$.

Because the triangles BFG and ACD are identical (Why?), it follows that the $\angle B = \angle A$.

27. **Theorem.** The adjacent angles DCA and DCB, which one straight line makes with another, are together equal to two right angles.



On C as a centre, with any radius, let the semicircle BDGA be described; then it is obvious that the angles BCD and DCA make up 180°, or twice 90°, that is, two right angles.

Or thus. From the point C draw CG at right angles to AB; then $\angle GCA + \angle GCB =$ two right angles;

$$\angle DCA = \angle GCA + \angle DCG;$$

adding $\angle DCB$ to these equals, we have

$$\angle DCA + \angle DCB = \angle GCA + \angle DCG + \angle DCB,$$
$$= \angle GCA + \angle GCB = 2 \text{ right angles.*}$$

Obs. The angle DCA is called the *supplement* of the angle DCB, and the angle DCG is called its *complement.*

Exercises on this Theorem.

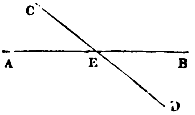
1. If $\angle BCD = 40°$, required its supplement DCA.
Ans. 140°; because $\angle DCA = 180° - 40° = 140°$.

2. If $\angle BCD = 35°$, required its complement DCG.
Ans. 55°; because $\angle DCG = 90° - 35° = 55°$.

3. What part of a right angle is 30°. *Ans.* $\frac{1}{3}$ part.

4. If the circumference of a circle be divided into 6 equal parts, how many degrees are there in each?
Ans. 60°; because the 6th part of $360° = 60°$.

5. What part of the whole circumference is an arc of 40°?
Ans. $\frac{1}{9}$ part; because 40° is just the $\frac{1}{9}$ part of 360°.



28. THEOREM. When two straight lines, AB and CD, cut each other, the opposite angles are equal. That is, $\angle AEC = \angle DED$, and $\angle CEB = \angle AED$.

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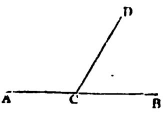
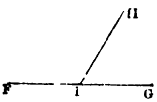
* The converse of this theorem is true, that is, if the straight line DC makes with CB and CA the sum of the angles DCB and DCA equal to two right angles, then CB and CA are in the same straight line.





From any point I in the straight line FG let IH be drawn, making the angle HIG equal to the angle DCB; then by the above theorem,

$$\angle HIG + \angle HIF = 2 \text{ right angles};$$

but by hypothesis $\angle DCB + \angle DCA = 2$ right angles;

$$\therefore \angle HIG + \angle HIF = \angle DCB + \angle DCA;$$

but $\angle HIG = \angle DCB$, therefore, taking away these equal angles, we have $\angle HIF = \angle DCA$.

Let the second figure now be placed upon the first, so that I shall fall upon C and IG upon CB; then IH will fall upon CD, because $\angle HIG = \angle DCB$; and also, since $\angle HIF = \angle DCA$, IF will fall upon CA; and therefore, ACB will coincide with the straight line FIG, that is, CB and CA are in the same straight line.

By the last theorem,

$$\angle AEC + \angle CEB = 2 \text{ right angles};$$

and also $\angle DEB + \angle CEB = 2$ right angles;

but things that are equal to the same thing are equal,

$$\therefore \angle AEC + \angle CEB = \angle DEB + \angle CEB.$$

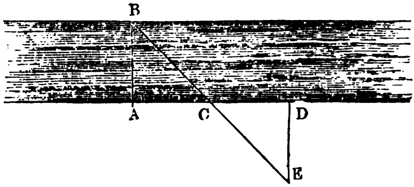
From these equals take away $\angle CEB$, and then

$$\angle AEC = \angle DEB.$$

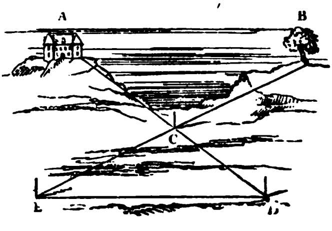
In the same manner it may be shown that $\angle CEB = \angle AED$.

Applications.

1. *To find the breadth* AB *of a river without crossing it.* Let the surveyor stand at A, directly opposite to an object B on the other side of the river. With the cross staff, or the theodolite, he lays down the line AD perpendicular to AB; he measures off any distance AC = CD, and lays down a line DE perpendicular to AD; he now places a staff at E in a line with the staff at C and the object B; then the distance DE being measured on the ground, will be the same as the breadth AB of the river. For in the triangles CDE and CAB, we have CD = CA; $\angle DCE = \angle ACB$ by the last theorem; and $\angle D = \angle A$, being both right angles; therefore, by Art. **17.**, the triangles are identical, and DE = AB.

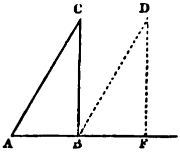


2. *To find the distance between two objects,* A *and* B, *inaccessible from the one to the other.* Take any station C, and measure the distances CB and CA; on the line CB produced or continued, *measure off* CE = CB, and



on the line CA produced, measure off $CD = CA$; then the distance DE being measured, will give the distance between the objects A and B. For in the triangles CDE and CAB, we have $CD = CA$; $CE = CB$, and $\angle DCE = \angle ACB$; therefore, by Art. 16., these triangles are identical, and $DE = AB$.

29. THEOREM. The exterior angle CBF of the triangle ABC is greater than either of the interior opposite angles A or C.



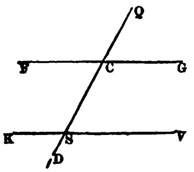
For let the triangle ABC slide along the line AF until the angle A comes to B, then it is plain that the vertex C will be found at some point D to the right of the side CB, and consequently the line BD must lie within the angle CBF; that is, the angle CBF must be greater than the angle DBF, or its equal CAB.

Producing the side CB, it may be shown by the same process of reasoning, that the $\angle CBF$ is greater than the $\angle C$.

Having established this theorem, the following axiom will readily be admitted.

Axiom. When the $\angle CBF$ is greater than the $\angle CAB$, the lines BC and AC meet in a point on the upper side of AF.

30. THEOREM. The two straight lines FG and KV are parallel when they make the same angles with a straight line DQ, that is, when $\angle FCS = \angle CSV$, or, what is the same thing, $\angle QCG = \angle CSV$.



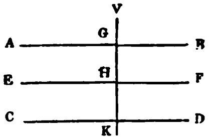
The lines cannot meet on the right side of QD, otherwise we should have a triangle formed; and then, by the preceding theorem, the exterior angle FCS would be *greater* than CSV, in the place of being *equal* to it; in like manner the lines cannot meet on the left side of QD; therefore the lines are parallel.

The angles FCS and CSV are called alternate angles; and QCG is called the exterior angle to the interior and opposite angle CSV.

31. Theorem. If QD cut two parallel straight lines FG and KV, the alternate angles FCS and CSV are equal, and also the exterior angle QCG is equal to the interior and opposite angle CSV. See the last figure.

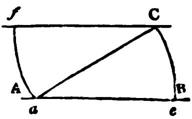
For if the angle FCS were greater than the angle CSV, the lines would meet and form a triangle on the right side of QD (Axiom, Art. **29.**); and on the contrary, if the angle FCS were less than the angle CSV, the lines would meet on the left side of QD; therefore, in order that FG and KV should neither meet on the one side nor the other, we must have ∠ FCS = ∠ CSV, and ∠ QCG = ∠ CSV.

32. Theorem. The straight lines AB and CD, which are parallel to the same line EF, are parallel to each other.



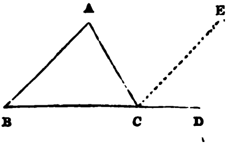
Let the straight line VK cut AB, EF, and CD. Then because AB and EF are parallel (Art. **31.**), ∠ VGB = ∠ VHF; and also because EF and CD are parallel, ∠ VKD = ∠ VHF; therefore, by equality, ∠ VGB = ∠ VKD; therefore, by Art. **30.**, AB and CD are parallel to each other.

33. Problem. Through a given point C to draw a straight line C*f* parallel to a given straight line AB.



From C draw any straight line C*d* to meet AB; from the point C (Art. **26.**) draw C*f*, making the ∠ *dcf* = ∠ *cde*; then C*f* will be parallel to AB by Art. **30.**

In practice parallel lines are most easily drawn by the triangular square described at Art. **10.** Place the edge AB of the square along the line KV (see fig. Art. **30.**), so that the edge AD shall pass through the given point C; then draw the line SQ; slide the square along the line SQ until the angle A of the instrument comes to C, then draw the line CG *along the edge* AB of the instrument, and it will be parallel to the given line KV. This method of drawing parallel lines is more simple, and not less accurate and expeditious, than that by the *parallel ruler.*



34. THEOREM. The exterior angle ACD of a triangle BCA, is equal to the sum of the interior and opposite angles B and A; and moreover the sum of the three angles of the triangle is equal to two right angles.

Let CE be drawn parallel to BA, then by Art. **31.**,

$$\angle \text{ECD} = \angle \text{B}; \text{ and } \angle \text{ECA} = \angle \text{A};$$

adding these equals together, we have

$$\angle \text{ECD} + \angle \text{ECA} = \angle \text{B} + \angle \text{A}, \text{ that is, } \angle \text{ACD} = \angle \text{B} + \angle \text{A}.$$

To each side of this equality add $\angle$ ACB, then

$$\angle \text{ACD} + \angle \text{ACB} = \angle \text{B} + \angle \text{A} + \angle \text{ACB};$$

But, by Art. **27.**, the left-hand side of this equality is equal to two right angles;

$$\therefore \angle \text{B} + \angle \text{A} + \angle \text{ACB} = \text{two right angles, or } 180^\circ.$$

Exercises and Applications.

1. If $\angle \text{A} = 25^\circ$, and $\angle \text{B} = 42^\circ$, required $\angle$ ACD.

Here $\angle \text{ACD} = 25^\circ + 42^\circ = 67^\circ$.

2. If the exterior $\angle \text{ACD} = 95^\circ$, and $\angle \text{A} = 36^\circ$; required the $\angle$ B.

Here $\angle \text{B} + \angle \text{A} = \angle \text{ACD}$, that is, $\angle \text{B} + 36^\circ = 95^\circ$; taking 36° from each side of the equality, $\angle \text{B} = 59^\circ$.

3. If $\angle \text{B} = 46^\circ$, and $\angle \text{A} = 84^\circ$, required the remaining $\angle$ ACB.

Here, $46^\circ + 84^\circ + \angle \text{ACB} = 180^\circ$; $\therefore \angle \text{ACB} = 50^\circ$.

4. The angles at the base of a triangle are 55° and 73° respectively; required the angle at the vertex. *Ans.* 52°.

5. Two angles of a triangle are 70° and 80°, required the remaining *angle.* Ans. 30°.

6. In a right-angled triangle the angle at the base is 27°, required the angle at the vertex. *Ans.* 63°.

7. Show that the angles opposite to the base and perpendicular of a right-angled triangle are acute.

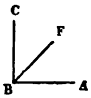
8. Show that the sum of the angles opposite to the base and perpendicular of a right-angled triangle is equal to 90°.

9. Show that the acute angles in a right-angled isosceles triangle are respectively 45°.

10. The angle at the vertex of an isosceles triangle is 50°, required the angles at the base. *Ans.* 65°.

11. To form an angle of 45°.

Draw BC perpendicular to BA, then $\angle ABC = 90^\circ$; from B draw BF, bisecting the $\angle ABC$ (Art. **21.**), then $\angle ABF = \frac{1}{2}$ of $90^\circ = 45^\circ$.



Bisecting the angle of 45° will give the angle of $22\frac{1}{2}^\circ$, and so on.

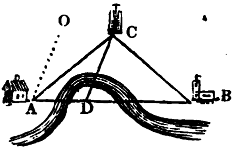
12. To form an angle of 60°, 30°, 15°.

Construct an equilateral triangle ABC (fig. Art. **20.**), then the angles A, B, and C, will be all equal to each other, and therefore $\angle A = \frac{1}{3}$ of $180^\circ = 60^\circ$.

Bisecting $\angle A$ we shall form an angle of 30°; and bisecting this latter angle we shall form an angle of 15°, and so on.

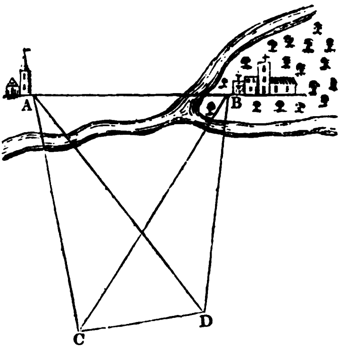
13. The distances of three objects, A, B, and C, from each other, are given, *viz.* $AB = 12$ miles, $BC = 7$ miles, and $AC = 8$ miles. At a station D, in a line with the objects A and B, the surveyor found the angle $BDC = 60^\circ$. Required the distance of the object C.

Construct the triangle ABC; from A draw AO, making the angle $BAO = 60^\circ$; from the point C draw CD parallel to AO, then $\angle CDB = \angle BAO = 60^\circ$; therefore D is the station; then DC being measured from the scale will give the required distance $= 5{\cdot}3$ miles.



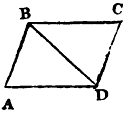
14. Wishing to determine the distance between two inaccessible objects A and B, I measured a base line $CD = 150$ yards. At the station D, I found the $\angle CDA = 45^\circ$, and the $\angle ADB = 22\frac{1}{2}^\circ$; then going to the station C, I found the $\angle DCB = 60^\circ$, and the $\angle BCA = 45^\circ$. Required the distance between the objects A and B.

From the scale of equal parts set off $CD = 150$; from D draw the lines DA and DB, making the $\angle CDA = 45^\circ$, and the $\angle ADB = 22\frac{1}{2}^\circ$; from C draw the lines CB and CA, making the $\angle DCB = 60^\circ$, and the $\angle BCA = 45^\circ$; then the lines DB and CB will intersect in the station B, and DA and CA in the station A; join A and B, and AB measured from the scale will give the required distance in units of yards = 158 yards nearly.



THEOREMS AND PROBLEMS RELATIVE TO PARALLELOGRAMS AND OTHER FOUR-SIDED FIGURES.

35. THEOREM. The opposite sides of a parallelogram are equal to each other; moreover the diagonal DB divides it into two equal parts, and the opposite angles are equal to each other.



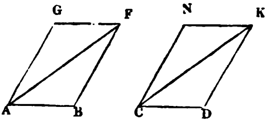
Because ABCD is a parallelogram, BC is parallel to AD, and therefore, by Art. **31.**, the alternate angles CBD and ADB are equal; in like manner the alternate angles CDB and ABD are also equal. Then in the two triangles CBD and ADB, we have the side BD common to both, $\angle CBD = \angle ADB$, and $\angle CDB = \angle ABD$; therefore, by Art. **17.**, these triangles are identical, and consequently $CB = AD$, $CD = BA$, and $\angle C = \angle A$.

Conversely. If CB = AD, and CD = BA, then ABCD is a parallelogram. For, by Art. **19.**, the triangles CBD and ADB will be identical, and ∠ CBD = ∠ ADB; therefore, by Art. **30.**, BC is parallel to AD; and in like manner DC is parallel to AB; that is, ABCD is a parallelogram.

36. THEOREM. If AD and BC be two equal parallel lines, then the lines AB and DC, joining the corresponding extremities, are also equal and parallel.

For in the triangles CBD and ADB, ∠ CBD = ∠ ADB, CB = DA, and DB is common to both triangles; therefore, by Art. **16.**, they are identical, and AB = DC, &c.

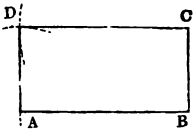
37. THEOREM. The parallelograms ABFG and CDKN are identical when AB = CD, BF = DK, and ∠ ABF = ∠ CDK.



The triangles ABF and CDK are identical (Art. **16.**); and also the triangles AGF and CNK. (Arts. **35.** and **19.**)

Let ABFG be placed upon CDKN, so that AB shall coincide with CD, then BF will fall on DK, and the point F on K; and the triangle FAG will exactly cover the triangle KCN; that is, the parallelograms are identical.

38. PROBLEM. To construct a rectangle ABCD, having given the two sides AB and BC which contain it; viz. AB = 36, and BC = 18.

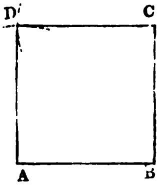


From a scale of equal parts take off AB = 36; draw BC perpendicular to AB, and from the same scale take BC = 18; on C as a centre with AB as a radius sweep an arc D; on A as a centre with BC as a radius sweep an arc cutting the former in the point D; draw the lines AD and CD; then ABCD will be the rectangle required. For, since DC = AB, and AD = BC, it follows by Art. **35.** that ABCD will be a parallelogram; and moreover as ∠ B is a right angle, it will also be a rectangle.

Cor. All the angles of a rectangle are right angles.

39. Problem. To construct a square upon the given line AB.

Draw BC perpendicular to AB, and take BC = AB; on A and C as centres, and with AB as a radius, describe arcs cutting each other in the point D; draw the lines AD and CD; then ABCD will be the square required. Since, by construction, the opposite sides of the figure are equal, Art. **35.**, it will be a parallelogram; and as BC = AB, and $\angle$ B = a right angle, it will also be a square.

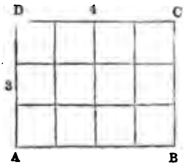


Cor. A square is a four-sided figure having all its sides equal, and all its angles right angles.

Applications.

1. *A unit of surface.* In order to compare surfaces with each other, it is necessary that we should fix upon some unit of surface. The unit selected for this purpose is the space included by the lines forming a square. Thus if the side AB, in the last figure, be a linear inch, then the square ABCD will contain an inch of surface, or 1 square inch; if AB be a linear foot, then the square ABCD will contain a foot of surface, or 1 square foot; and so on. The student will therefore not fail to observe, that a unit of surface is essentially different from a simple linear unit or unit of length.

2. *To find the units of surface in a rectangle* ABCD. Let AB = 4 inches, and AD = 3 inches. Here, by drawing the parallel lines through the divisions in the sides AB and AD, we evidently divide the rectangle into units of square inches. The number of square units in each horizontal row will obviously be equal to the number of linear units in the side AB; and the number of these rows will be equal to the num-



ber of linear units in the side AD; therefore the number of square units in the whole rectangle will be the units in AB multiplied by the units in AD, that is, the units of surface in the rectangle ABCD = AB × AD. The units of surface in a figure is called its *area*. In the example proposed, the area $= 4 \times 3 = 12$ square inches. Hence the rectangle ABCD is symbolically expressed by AB . AD, and the square ABCD (Art. **39.**) by AB . BC or AB^2. Hence we have the following rule for finding the area of a rectangle.

Rule.—Multiply the length by the breadth, and the product will be the area of the rectangle.

Ex. (1.) Required the area of a rectangle whose length is 9 ft. and breadth 7 ft.

Here, area $= 9 \times 7 = 63$ sq. ft.

(2.) The side of a square is 6 inches, required its area. *Ans.* 36 sq. in.

(3.) Show that there are 9 square feet in a square yard.

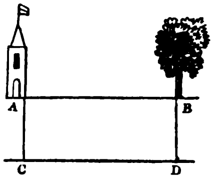
(4.) Show that there are 144 square inches in a square foot.

(5.) Show that the square described upon the whole line is four times the square described upon one half the line.

(6.) Show that a rectangle 1 foot long, and 1 inch broad, contains one-twelfth of a square foot.

For additional exercises, see PROBLEMS IN MENSURATION.

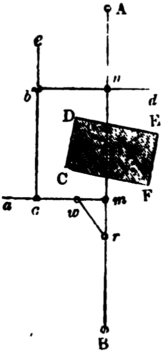
3. *To measure the distance between two objects* A *and* B, *inaccessible the one from the other, by means of the cross staff.*



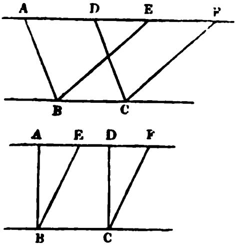
Lay down on the ground a straight line CD parallel to AB, by taking any distance BD = AC; with the cross staff, Art. **10.**, find the point C where CA forms a right angle with the line CD; after the same manner find the point D; then the distance CD being measured will give the required distance AB. For ABDC will be a parallelogram, and therefore the opposite sides AB and CD will be equal.

4. *To continue the straight line* B *m beyond the impediment* CDEF.

From the point *m* (with the cross staff, or by the method explained at Art. 48.) lay down the perpendicular line *m a*; on this line measure off *m c* = 1 chain, or any other convenient distance; lay down the line *c e* perpendicular to *m a*; from any point *b*, beyond the obstruction, lay down the line *bd* perpendicular to *c e*, and measure off *b n* = *c m*; from *n* lay down *n* A perpendicular to *n b*; then *n* A will be the continuation of the straight line B *m*, and *c b* being measured will give the distance between *m* and *n*. For *m c b n* is evidently a rectangle, and therefore *m n* = *c b*; and as the angles *b n* A, *b n m*, *c m n*, and *c m* B are all right angles, *n* A and B *m* will be in the same straight line with *m n*.



40. THEOREM. The parallelograms BCDA and BCFE, between the same parallels BC and AF, and on the same base BC, are equal.



By Art. 35. AD = BC, and also EF = BC;

∴ AD = EF, because they are equal to the same thing, *viz.* BC. From AF take away AD, and DF will remain; again, from AF take away EF, and AE will remain; therefore DF = AE, because we have taken equal lines away from the same line. Now in the triangles DCF and ABE, we have DC = AB, CF = BE, and DF = AE; therefore, by Art. 19., these triangles are identical.

From the whole figure BCFA, first take away the triangle DCF, and the *parallelogram* BCDA will remain; then from the same figure BCFA take away the triangle ABE, and the parallelogram BCFE will remain;

∴ the parallelogram BCDA = the parallelogram BCFE, because equal spaces are here taken away from the same space.

Application of this theorem.

To convert the parallelogram BCFE (see the last fig.) into the rectangle BCDA. From the parallelogram BCFE cut off the triangle CDF, and place it on BAE; then BCFE will be converted into the rectangle BCDA.

From this theorem it follows that the area of a parallelogram is equal to the area of a rectangle having the same base and perpendicular height; hence we have the following rule for finding the area of a parallelogram.

Rule.—Multiply the linear units in the base by the linear units in the perpendicular, and the product will be the square units in the surface of the parallelogram.

Ex. 1. Required the area of the parallelogram BCFE, when the base BC = 12 ft., and the perpendicular height CD = 9 ft.

Here the area $= 12 \times 9 = 108$ square feet.

2. Required the area of a parallelogram whose base is 5·6 ft., and perpendicular 3·2 ft. *Ans.* 17·92 *sq. ft.*

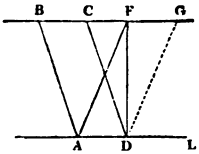
3. Required the side of a square, which shall contain the same surface as a parallelogram whose base is 16 yds., and perpendicular height 9 yds.

Let $x =$ the number of feet in the side,
$\therefore x \times x$, or $x^2 =$ the number of sq. ft. in the square,
and $16 \times 9 = 144 =$ the number of sq. ft. in the parallelogram,
$\therefore$ by the question, $x^2 = 144$; $\therefore x = 12$ ft.

4. Required the breadth of a rectangle 27 ft. long, which shall contain the same surface as the parallelogram of *Ex.* 1. *Ans.* 4 *ft.*

5. The area of a parallelogram is 204 sq. ft., and its base is 17 ft.; required its perpendicular height. *Ans.* 12 *ft.*

41. THEOREM. If a parallelogram ADCB and a triangle ADF stand between the same parallels AL and BG, and on the same base AD, the triangle will be half the parallelogram.

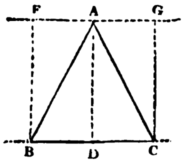


For, draw DG parallel to AF; then, by Art. **35.**, the triangle ADF = half the parallelogram ADGF; but the parallelogram ADGF = the parallelogram ADCB; therefore the triangle ADF is also = half the parallelogram ADCB.

Cor. Triangles upon the same base and between the same parallels are equal.

Application of this Theorem.

1. To show that the rectangle BCGF contains double the surface of the triangle BCA. Let AD be perpendicular to the base BC; then if ACG be cut off from the rectangle, it will exactly fit the part ADC; and in like manner ABF will exactly fit the part ADB; thereby showing that the rectangle is double the triangle.



From this theorem it follows that the area of a triangle is equal to half the area of a rectangle having the same base and perpendicular height: hence we have the following rule for the area of a triangle.

Rule. — Multiply the linear units in the base by the linear units in the perpendicular height, and half the product will be the square units in the surface of the triangle.

Ex. (1.) In the triangle BCA, the base BC = 7 ft., and the perpendicular height AD = 8 ft., required the area.

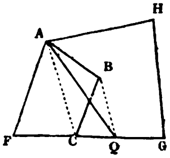
Here, area rectangle BCGF $= 7 \times 8$;

but the triangle BCA contains half this surface;

$$\therefore \text{ area triangle BCA} = \frac{7 \times 8}{2} = 28 \text{ sq. ft.}$$

(2.) The base of a triangle is 25 in., and the perpendicular height is 22 in., required the area. *Ans.* 275 *sq. in.*

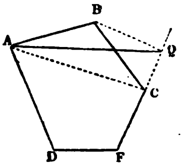
2. To convert a crooked hedge ABC into a straight one, without altering the size of the two fields ABCF and ABCGH.



Join AC, and draw BQ parallel to it; then join AQ, and it will be the direction of the hedge required.

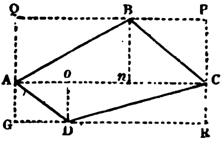
For since the triangle AOQ is equal to the triangle ACB (*Cor.* Art. **41.**), the area AQF will be equal to the area ABCF.

3. Let DABCF be the section of a railway cutting, it is required to convert it into the equivalent area DAQF.



Here the process is precisely the same as in the preceding problem.

42. Theorem. The trapezium ADCB is half the circumscribing rectangle GRPQ, having its side GR parallel to the diagonal AC.



By Art. **41.**,

The triangle ABC = half the rectangle AOPQ;

The triangle ADC = half the rectangle ACRG;

therefore, by adding these equals together, we have

The trapezium ADCB = half the rectangle GRPQ.

Application of this Theorem.

From this theorem we derive the following rule for finding the area of a trapezium.

Rule. Multiply the diagonal by the sum of the perpendiculars let fall upon it, and half the product will be the area of the trapezium.

Because $GQ = Do + Bn$, therefore by the last theorem,

$$\textit{Area } ADCB = \frac{GR \times GQ}{2} = \frac{AC \times (Do + Bn)}{2}.$$

Ex. 1. The diagonal AC of a trapezium is 26 ft., the perpendicular Bn = 8 ft., and Do = 6 ft., required the area.

Here the sum of the perpendiculars, or GQ = 8 + 6 = 14 ft.

∴ Area circumscribing rectangle GRPQ = 26 × 14;

but the area of the trapezium ADCB is half this rectangle;

$$\therefore \text{ Area ADCB} = \frac{26 \times 14}{2} = 182 \text{ sq. ft.}$$

$$\text{Or thus: Area triangle ABC} = \frac{26 \times 8}{2} = 104 \text{ sq. ft.}$$

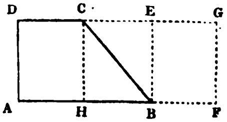
$$\text{,, ,, ADC} = \frac{26 \times 6}{2} = 78 \text{ sq. ft.}$$

$$\therefore \text{ Area ADCB} = 104 + 78 = 182 \text{ sq. ft.}$$

2. Required the area of a trapezium, whose diagonal is 18 yds., and the perpendiculars upon it, 5 and 4 yds.

Ans. 81 *sq. yds.*

43. THEOREM. The trapezoid ABCD, having the sides AB and DC parallel, is equal to half the parallelogram AFGD, having the same altitude, and whose base AF is the sum of the parallel sides AB and DC.



Draw CH and BE parallel to AD or FG; then the triangle CBE will be identical with the triangle CBH; and because BF = CD, the parallelogram BFGE will be identical with the parallelogram AHCD; therefore the trapezoid CGFB will be identical with the trapezoid ABCD, since the parts of the one exactly fit the parts of the other. Hence it follows that the trapezoid ABCD will be half the parallelogram AFGD.

Application of this Theorem.

From this theorem we derive the following rule for finding the area of a trapezoid.

RULE. Multiply the sum of the two parallel sides by the perpendicular distance between them, and half the product will be *the area of the* trapezoid.

Ex. 1. Required the area of a trapezoid ABCD, when $AB = 6$ ft., $DC = 4$ ft., and the perpendicular distance $HC = 5$ ft.

$$\text{Here the sum of the parallel sides } AF = 6 + 4 = 10;$$

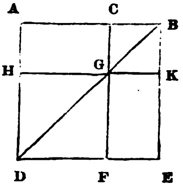
$$\therefore \text{ Area rectangle } AFGD = 10 \times 5;$$

but the area of the trapezoid ABCD is half this rectangle;

$$\therefore \text{ Area } ABCD = \frac{10 \times 5}{2} = 25 \text{ sq. ft.}$$

2. The parallel sides of a trapezoid are 9 and 10 ft. respectively, and the perpendicular distance between them is 8 ft., required the area. *Ans.* 76 *sq. ft.*

44. THEOREM. The square of the whole line DE is equal to the squares of the two parts DF and FE, together with twice the rectangle of those parts. That is, $DE^2 = DF^2 + FE^2 + 2\,DF \cdot FE$.



Let DEBA be the square on DE, and DFGH the square on DF; then producing FG to C, and HG to K, the part CBKG will be the square on CB or FE (Why?); and the rectangle ACGH will be identical with the rectangle FEKG (Art. **37.**); but this rectangle is contained by the lines FE and EK, or FE and DF. Now the whole square DEBA is made up of the square DFGH, the square CBKG, together with the two rectangles ACGH and FEKG;

$$\therefore DE^2 = DF^2 + FE^2 + 2\,DF \cdot FE.$$

This theorem may also be readily established by algebra. For this purpose, let $DF = a$, $FE = b$, then,

$$(a+b)^2 = a^2 + b^2 + 2\,ab; \text{ that is,}$$

$$(DF + FE)^2, \text{ or } DE^2 = DF^2 + FE^2 + 2\,DF \cdot DE \ldots (1).$$

After the same manner we have, putting $a = DE$,

$$(a-b)^2 = a^2 + b^2 - 2\,ab; \text{ that is,}$$

$$(DE - FE)^2, \text{ or } DF^2 = DE^2 + FE^2 - 2\,DE \cdot FE \ldots (2);$$

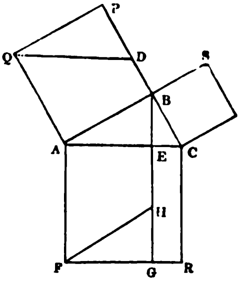
that is, the square described on the difference of two lines is equal to the sum of the squares of those lines diminished by twice their rectangle.

$$\textit{And also, } a^2 - b^2 = (a+b)(a-b) \ldots (3);$$

that is, the difference of the squares of any two lines is equal to the rectangle on their sum and difference.

THE FORTY-SEVENTH PROPOSITION OF EUCLID, AND CERTAIN PROPERTIES OF RIGHT-LINED FIGURES DEPENDING UPON IT.

45. Theorem. The square described upon the hypotenuse AC of a right-angled triangle ABC is equal to the sum of the squares described upon the other two sides; that is, $AC^2 = AB^2 + CB^2$.



Let ACRF be a square described on the hypotenuse AC. On the line CB produced take BP = BA; draw PQ parallel to BA, and AQ parallel to BP: then because the ∠ ABC is a right angle, the ∠ ABP will also be a right angle; therefore ABPQ will be a square; in the same manner let the square BCKS be constructed. Draw BG parallel to AF; QD to AC; and FH to AB; then

$$\angle\, CAF = \angle\, BAQ, \text{ being each a right angle;}$$

to these equals add ∠ CAB, then

$$\angle\, CAF + \angle\, CAB = \angle\, BAQ + \angle\, CAB; \therefore \angle\, BAF = \angle\, CAQ.$$

Now in the parallelograms ABHF and ACDQ, we have AF = AC; AB = AQ; and the angles included by these sides also equal, that is, ∠ BAF = ∠ CAQ; therefore, by Art. **37.**, these parallelograms are identical. But by Art. **40.** the square ABPQ = the parallelogram ACDQ; and the rectangle AFGE = the parallelogram ABHF; and since things that are equal to the same thing are equal;

$$\therefore \text{ the square } ABPQ = \text{the rectangle } AFGE.$$

In precisely the same way, the other square CBSK is proved equal to the other rectangle CRGE; therefore the sum of the two squares ABPQ and CBSK are together equal to the two rectangles AFGE and CRGE, or the square ACRF.

Cor. 1. The hypotenuse AC is greater than either of the sides AB or BC: because AC^2 is greater than AB^2, therefore *AC will be greater than AB.*

Cor. 2. From each side of the equality, $AB^2 + CB^2 = AC^2$, take away CB^2, $\therefore AB^2 = AC^2 - CB^2$.

Applications of the preceding Theorem.

Ex. 1. The two sides of a right-angled triangle are 8 and 6 ft. respectively, required the hypotenuse.

Let $x =$ the units in the hypotenuse, then

$$x^2 = 8^2 + 6^2 = 100;$$

taking the square root of each side of the equality,

$$x = \sqrt{100} = 10.$$

2. The sides of a right-angled triangle are 16 and 12 ft. respectively, required the hypotenuse. *Ans.* 20 *ft.*

3. When the hypotenuse is 25, and one of the sides is 15, what must be the other side?

Let $x =$ the other side, then

$$x^2 + 15^2 = 25^2;$$

taking 15^2 from each side of this equality,

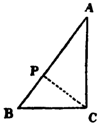
$$x^2 = 25^2 - 15^2 = 400;$$

taking the square root of each side,

$$x = \sqrt{400} = 20.$$

4. When the hypotenuse is 30, and one of the sides is 24, what must be the other side? *Ans.* 18.

46. Given the base BC, and perpendicular AC, of the right-angled triangle ABC, to find the perpendicular CP upon the hypotenuse.



Ex. 1. Let $BC = 21$, and $AC = 28$, then,

$$AB^2 = 21^2 + 28^2; \therefore AB = 35.$$

Now the area of the triangle ABC may be found in two ways.

$$\text{1st, Area } ABC = \frac{21 \times 28}{2}; \text{ 2nd, Area } ABC = \frac{35 \times CP}{2};$$

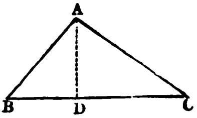
but things that are equal to the same thing are equal,

$$\therefore \frac{35 \times CP}{2} = \frac{21 \times 28}{2};$$

from this equation we find, $CP = 16{\cdot}8$.

2. Required CP, when $BC = 24$, and $AC = 32$. *Ans.* 19·2.

47. Given the three sides of the triangle ABC to find; 1st, the segment of the base BD; 2d, the perpendicular AD; 3d, the area of the triangle.



Ex. 1. Let $BC = 20$, $BA = 10$, and $AC = 12$.

Put $x = BD$, then $DC = 20 - x$.

Now, from the right-angled triangles ADB and ADC, we can find two independent expressions for AD^2.

See Cor. 2,

$$AD^2 = 10^2 - x^2, \text{ and also } AD^2 = 12^2 - (20 - x)^2.$$

But these expressions for the same thing must be equal.

$$\therefore\ 12^2 - (20 - x)^2 = 10^2 - x^2.$$

From this equation we readily find $x = 8·9$, which is the segment BD.

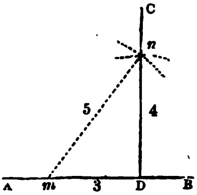
To find the perpendicular, we have $AD^2 = 10^2 - 8·9^2$, and $\therefore$ $AD = 4·55$.

$$\text{Hence the area of the triangle} = \frac{20 \times 4·55}{2} = 45·5.$$

2. Required the same as in the last example, when $AC = 6$, $BA = 4$, and $CB = 5$.

Ans. $BD = ·5$, $AD = 3·96$, and area $= 9·9$.

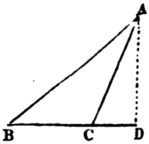
48. To raise a perpendicular on the ground by means of the chain or tape line.



From the given point D, in the chain line AB, let it be required to lay down the perpendicular DC.

Measure off $D\,m = 30$ links; with one extremity of the chain at m extend 90 links along mnD, then if a staff be put at n, making $Dn = 40$ links, and $\therefore$ $mn = 50$ links, the line Dn produced will be the perpendicular required.

For mDn will be a right-angled triangle, since $3^2+4^2=5^2$; and any multiple of 3, 4, and 5, will have the same property.



49. THEOREM. In an obtuse-angled triangle, the square of the side AB subtending the obtuse angle, is greater than the sum of the squares of the other two sides, BC and AC, by twice the rectangle of the base BC and the distance CD of the perpendicular from the obtuse angle. That is, $AB^2=BC^2+AC^2+2\,BC\,.\,CD$.

By Art. **44.** we have,

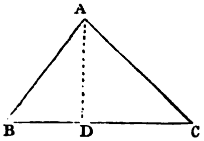
$$BD^2=BC^2+CD^2+2BC\,.\,CD\,;$$

adding AD^2 to these equals, we have,

$$BD^2+AD^2=BC^2+CD^2+AD^2+2BC\,.\,CD\,;$$

$$\text{but } BD^2+AD^2=AB^2;\ \text{and } CD^2+AD^2=AC^2\,;$$

$$\therefore\ AB^2=BC^2+AC^2+2BC\,.\,CD.$$



50. THEOREM. When the side AB subtends an acute angle, then

$$AB^2=BC^2+AC^2-2\,BC\,.\,CD.$$

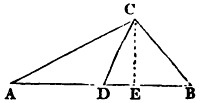
By Art. **44.** equality (2),

$$BD^2=BC^2+CD^2-2\,BC\,.\,CD\,;$$

adding AD^2 to these equals, and proceeding as in the last theorem, we have

$$BD^2+AD^2=BC^2+CD^2+AD^2-2\,BC\,.\,CD\,;$$

$$\therefore\ AB^2=BC^2+AC^2-2\,BC\,.\,CD.$$



51. THEOREM. In any triangle ABC, if CD be drawn from the vertex to the middle of the base, then we shall have,

$$AC^2+CB^2=2\,AD^2+2\,CD^2.$$

Draw CE perpendicular to AB, then from the triangles ADC and DBC, by the two preceding theorems, we have,

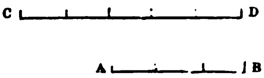
$$AC^2=AD^2+CD^2+2\,AD\,.\,DE$$
$$CB^2=BD^2+CD^2-2\,BD\,.\,DE.$$

Adding these equalities together, and observing that AD = BD, we have,

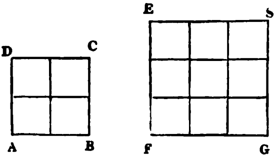
$$AC^2 + CB^2 = 2\,AD^2 + 2\,CD^2.$$

RATIO OF LINES AND SURFACES: SIMILAR TRIANGLES.

52. The ratio of two lines, or indeed quantities of any kind, is their relative magnitude. Thus, if the line CD contains five units, and AB three units, then CD will be five times the third of AB, or, as the ratio is usually expressed, $\frac{CD}{AB}=\frac{5}{3}$; thus it will be seen that the ratio of CD to AB is the number of times that AB is contained in CD.

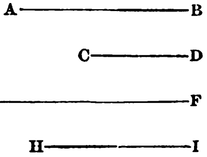


In like manner if the surface of the square FS contains nine square units, and the square AC four square units, then the surface FS will be nine times the fourth of the surface AC, or $\frac{FS}{AO}=\frac{9}{4}$.



When we have an equality of ratio, the terms form a proportion; thus, if we have $\frac{CD}{AB}=\frac{5}{3}$, we shall also have the proportion CD : AB :: 5 : 3. Or generally, if the ratio of AB to CD is equal to the ratio of EF to HI, we shall have,

$$\frac{AB}{CD}=\frac{EF}{HI}\ \ldots\ (1)$$



Expressing this equality of ratio in the usual *form* of a proportion, we have,

$$AB : CD :: EF : HI\ \ldots\ (2),$$

which we express in language by saying,

AB is to CD, as EF is to HI.

From the definition (1) of this proportion, we readily find by multiplication,

$$AB \cdot HI = CD \cdot EF \quad . \; . \; . \; (3);$$

that is, *in any proportion the product of the extreme terms is equal to the product of the middle or mean terms.*

From this last equality we have by division,

$$\frac{AB}{EF} = \frac{CD}{HI}, \text{ or } AB : EF :: CD : HI;$$

hence it appears that if four terms are in proportion, as in eq. (2), then they will also be in proportion when we interchange the places of the two middle terms. This is called *alternation* of the terms.

Similarly we have, CD : AB :: HI : EF;

where the terms of the proportion in (2) are inverted.

Again, let A : B :: C : D, and also A : B :: E : F, then C : D :: E : F. Because, from the first proportion we have, $\frac{A}{B} = \frac{C}{D}$, and from the second $\frac{A}{B} = \frac{E}{F}$, then by equality $\frac{C}{D} = \frac{E}{F}$, and

∴ C : D :: E : F.

Various other properties may be readily deduced.*

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* It is highly desirable that the student should know something about the doctrine of *limits*, and its application to the proof of certain geometrical theorems. Let $a + cx = b + ey$ be an equation such that a and b are constant, but cx and ey admit of being diminished to any extent we please, by decreasing the variable quantities x and y; then we shall have $a = b$, and $cx = ey$.

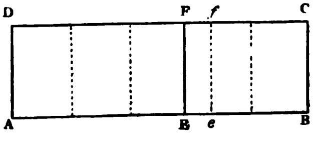
For we have, by transposition, $cx - ey = b - a$;

now if b is not equal to a, let the difference be some constant quantity r, then $cx - ey = r$. From this equation it appears, that the difference of cx and ey cannot become less than the fixed quantity r, which is contrary to the hypothesis; therefore we must have $a = b$, and $cx = ey$. This constitutes the principle of the *method of limits*. Here, by diminishing x and y, the values of a and b approximate to each other, so as to render their difference less than any quantity which can be assigned. The proposed equation taken at its *limit* gives $a = b$.

The rule given at page 34. may be more rigorously established by the application of this principle.

3. **THEOREM.** Let CA be divided into any number of al parts, $Ce=er=rs=\ldots$, and let eh, rm, sp, ... be

he rectangles ABCD *and* AEFD, *having the same altitude* AD, *are to each r as their bases* AB *and* AE.



irst let the bases be commenble, or exact multiples of e other line; and suppose AB e divided into five equal parts, that AE contains three of m; from each point of the ision erect a perpendicular to base, then five equal rectangles will thus be formed; but ABCD contain five of them, and AEFD three of them,

$$\therefore \frac{ABCD}{AEFD}=\frac{5}{3}, \text{ or } \frac{AB}{AE};$$

in the form of a proportion,

$$ABCD : AEFD :: AB : AE.$$

t is obvious that precisely the same reasoning may be applied, whatmay be the ratio of AB to AE.

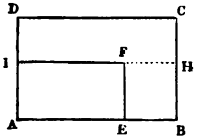
econd, let the bases be incommensurable: suppose AB to be divided any number of equal parts, and let e be the point in the division est to E, then, because AB and Ae are commensurable,

$$\frac{Aef D}{ABCD}=\frac{Ae}{AB}, \text{ or } \frac{AEFD}{ABCD}+\frac{Eef F}{ABCD}=\frac{AE}{AB}+\frac{Ee}{AB};$$

because E$ef$F and Ee may be reduced as small as we please, whilst other quantities remain constant, we have by the principle of limits,

$$\frac{AEFD}{ABCD}=\frac{AE}{AB}.$$

he rectangles ABCD *and* AEFI *are to each other as the product of their s by their altitudes.*



roduce IF to H, then by the preceding rem,

$$\frac{ABHI}{AEFI}=\frac{AB}{AE}, \text{ and also } \frac{ABCD}{ABHI}=\frac{AD}{AI};$$

tiplying these equations together,

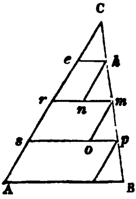
$$\frac{ABCD}{AEFI}=\frac{AB\times AD}{AE\times AI}.$$

et AI and AE be each taken unity, then AEFI will be a unit of ace, and

$$\therefore \text{the units of surface in } ABCD=AB\times AD,$$

re AB and AD are understood to mean the number of linear units coned in them. In this sense, therefore, we say that the area of a llelogram is equal to the product of the base by the perpendicular. a triangle is one-half the parallelogram having the same base and ude, the area of a triangle is measured by one-half the product of the *by the perpendicular.*

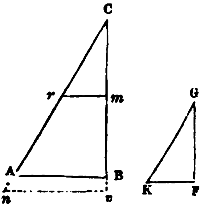
drawn parallel to AB, then CB will be divided into the same number of equal parts, $Ch = hm = mp = \dots$



Draw hn, mo, ... parallel to CA; then the triangles ceh and hnm will be identical; because of the parallel lines, $ehnr$ will be a parallelogram, and therefore $nh = re = ec$, $\angle mhn = \angle hce$, and $\angle hnm = \angle ceh$; therefore, Art. **17.**, $hm = ch$. In precisely the same way it may be shown that $mp = ch$, and so on.

Cor. Taking any of the points r in CA, it follows that Cr will be contained in CA as many times as Cm is contained in CB, that is,

$$\frac{CA}{Cr} = \frac{CB}{Cm}, \text{ or } CA : Cr :: CB : Cm.$$



54. THEOREM. Equiangular triangles are similar, or have their like sides proportional.

Let ABC and KFG be two equiangular triangles, having $\angle A = \angle K$, $\angle B = \angle F$, and $\therefore$ $\angle C = \angle G$, then will

$$CA : CB :: GK : GF,$$

For, take $cr = $ GK, and $cm = $ GF, and join the points r and m; then the triangle crm will be identical with GKF. Since $\angle r = \angle K$ or $\angle A$; therefore (Art. **30.**) rm is parallel to AB.

Let cr and rA be divided into any number of equal parts; now if lines be drawn, from the points of division, parallel to AB, then, by the last theorem, the line cm will be cut into the same number of parts that there are in cr, and CB into the same number of parts that there are in CA. Hence it follows that cr or GK will be contained in CA as many times as cm or GF is contained in CB; that is,

$$\frac{CA}{GK} = \frac{CB}{GF}, \text{ or } CA : GK :: CB : GF;$$

and by alternation, Art. **52.**, CA : CB :: GK : GF.*

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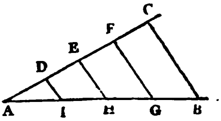
* *If the lines cr and* CA *are incommensurable, that is, if the parts into*

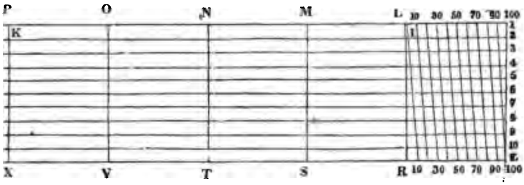
Applications of the preceding Theorem.

1. To divide a given line AB into any number of equal parts; let the number of parts be four for example.



From A draw any straight line AC, and with any opening of the compasses mark off the four equal parts AD, DE, EF, and FC; join the points C and B, and draw FG, EH, and DI parallel to CB; then AI = IH = HG = GB.

2. The Diagonal Scale. In this scale the equal divisions



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which cr is divided do not exactly divide the line CA, then the theorem may be established in the following manner:—

Let cr be divided into any number of equal parts, and measure off one of them from r towards A; let An be the portion in excess, then draw nv parallel to AB. Because Cn and cr are commensurable, we have, by the above theorem,

$$\frac{cn}{cr} = \frac{cv}{cm};$$

but $cn = \text{CA} + \text{A}n$, and $cv = \text{CB} + \text{B}v$, ∴ by substitution,

$$\frac{\text{CA} + \text{A}n}{cr} = \frac{\text{CB} + \text{B}v}{cm}, \text{ or } \frac{\text{CA}}{cr} + \frac{\text{A}n}{cr} = \frac{\text{CB}}{cm} + \frac{\text{B}v}{cm}.$$

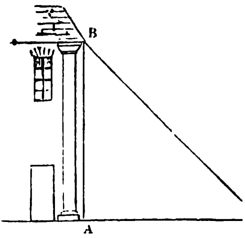
By increasing the number of divisions in cr, the excess An and Bv may be made as small as we please; therefore, $\frac{\text{A}n}{cr}$ and $\frac{\text{B}v}{cm}$ may be indefinitely decreased, and then we ultimately must have, (see note, Art. **52.**)

$$\frac{\text{CA}}{cr} = \frac{\text{CB}}{cm}.$$

This theorem may also be proved precisely after the method given in *the note to Art.* ***64.***

ER, RS, ST, &c., are supposed to contain 100 parts divided into ten parts, each part will contain 10 the diagonal lines being drawn from 10 to L, 20 t 20, &c., enable us to take off units; for exampl tance from K to I will be 401; and taken on the line 3 from PX to the diagonal L 10, will be 402, As a further illustration, let it be required to tak first extending the compasses from O to 50 will gi and for the 4 units, carry down the left point o passes on the vertical line OV until it comes t horizontal line; then extend the other point of the until it reaches to the intersection of the diagon with the horizontal line marked 5, and it will gi tance 354 required.

3. To find the height AB of an object by m mirror placed horizontally at C. Let the surveyor be at D, when he observes the reflection of the top of the object B; then as the angle of incidence of a ray of light is always equal to the angle of reflection, it follows that $\angle ACB = \angle DCE$, and the triangles ACB and DC similar; hence we have

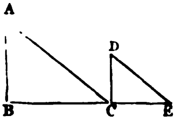


$$DC : DE :: CA : AB; \quad \therefore AB = \frac{DE \cdot CA}{DC}.$$

For example, suppose the surveyor finds by me CA = 100 ft., DC = 6 ft., and the height of the eye then,

$$\text{The height of the object } AB = \frac{5 \times 100}{6} = 83\tfrac{1}{3}$$

4. To find the height of a tower by means of t



of a pole. Let AB be the height of the tower and BC its shadow; DC the height of the pole and CE its shadow; then as the sun's rays AC and DE (forming the extremities of the shadows) are parallel to each other, it follows that ∠BCA = ∠CED, and therefore the triangles BCA and CED are similar.

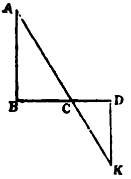
$$\therefore \text{CE} : \text{CD} :: \text{BC} : \text{AB}; \ \therefore \text{AB} = \frac{\text{CD} \,.\, \text{BC}}{\text{CE}}.$$

(*Ex.* 1.) The shadow cast by a pole 10 ft. long is 7 ft., required the height of a tower which at the same time casts a shadow of 140 ft.

In this case we have,

$$7 : 10 :: 140 : \text{AB} = 200 \text{ ft.}$$

(2.) Required the same as in the last example, when CD =5 ft., CE=4 ft., and BC=64 ft. *Ans.* 80 *ft.*



5. To find the distance BA of an object A without approaching it. By means of the cross staff or theodolite, lay down the line BD at right angles to BA; take BC and CD any convenient distances, and place a staff at C; lay down DK at right angles to BD, and put a staff at K so that it may be in the same straight line with C and A, and then measure DK.

Because ∠BCA = ∠DCK, and ∠B = ∠D, being each right angles, the triangles BCA and CDK are similar.

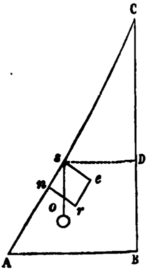
$$\therefore \text{CD} : \text{DK} :: \text{CB} : \text{AB}; \ \therefore \text{AB} = \frac{\text{DK} \,.\, \text{CB}}{\text{CD}}.$$

(*Ex.* 1.) Let BC=40 ft., DC=20 ft., and DK=60 ft., then,

$$20 : 60 :: 40 : \text{AB} = 120 \text{ ft.}$$

(2.) Required the distance AB, when BC=4 chains, DC= 1 chain, and DK=3 chains. *Ans.* 12 *chains.*

6. To measure the height of an inaccessible object CB by means of *the geometrical* square.



The geometrical square consists of a square frame, $n\,s\,e\,r$, having the edges, $s\,e$, $e\,r$, and $r\,n$, divided each into 100 equal parts; a plummet, $s\,o$, hangs from the corner s; and on the edge $n\,s$ are two sights, which enable the surveyor to direct the edge $n\,s$ towards the summit C of the object. By this instrument the elevation of an object is readily determined by the property of similar triangles.

CASE I. When the plummet cuts the side $r\,n$.

Let s D be parallel to the horizon, and D B the height of the instrument from the ground. Measure the base line s D and the height D B of the instrument; direct the sights n and s to the top C of the object, and then observe the number of parts in $n\,o$.

The triangles $n\,o\,s$ and s D C are similar, for since $s\,o$ is parallel to C B, the $\angle\, n\,s\,o = \angle\, s\,\mathrm{C\,D}$, and $\angle\, n = \angle\, \mathrm{D}$, being each of them right angles.

$$\therefore n\,o : n\,s :: s\mathrm{D} : \mathrm{CD} = \frac{n\,s\,.\,s\mathrm{D}}{n\,o},$$

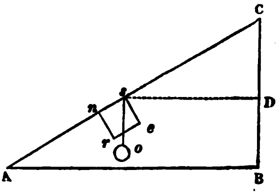
$$\text{And the height } \mathrm{CB} = \frac{n\,s\,.\,s\mathrm{D}}{n\,o} + \mathrm{DB}.$$

Ex. 1. Let $s\mathrm{D} = 66$ ft., $n\,o = 20$, and $\mathrm{DB} = 5$ ft. Here we have, $20 : 100 :: 66 : \mathrm{CD} = 330$ ft.; and $\mathrm{CB} = 330 + 5 = 335$ ft.

2. Required the height C B, when $s\mathrm{D} = 60$ ft., $n\,o = 50$ ft., and the height of the instrument $\mathrm{DB} = 6$ ft. *Ans.* 126 *ft.*

CASE II. When the plummet cuts the edge $r\,e$.

In this case the triangles $o\,e\,s$ and s D C are similar, and then we find, $\mathrm{CB} = \dfrac{e\,o\,.\,s\mathrm{D}}{s\,e} + \mathrm{DB}$.

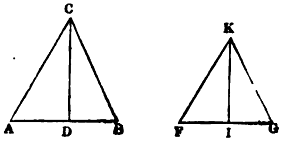


Ex. 1. Let $s\mathrm{D} = 66$ ft., $e\,o = 60$, *and* $\mathrm{DB} = 5$ *ft.*

ere, $100 : 60 :: 66 : \mathrm{CD} = 39{\cdot}6$; and $\therefore\ \mathrm{CB} = 39{\cdot}6 + 5$ 4·6 ft.

Required CB when $s\,\mathrm{D} = 100$ ft., $e\,o = 80$, and $\mathrm{DB} =$ *Ans.* 85.

5. **THEOREM.** The areas of equiangular or similar tries are to each other as squares of their like s.



et ABC and FGK be the ıngular triangles; let fall erpendiculars CD and KI, the triangles ACD and FKI will also be equiangular (Why?)

$$\therefore \frac{\mathrm{AB}}{\mathrm{FG}} = \frac{\mathrm{AC}}{\mathrm{FK}}, \text{ and } \frac{\mathrm{CD}}{\mathrm{KI}} = \frac{\mathrm{AC}}{\mathrm{FK}};$$

plying these equals together, we have

$$\frac{\mathrm{AB}\,.\,\mathrm{CD}}{\mathrm{FG}\,.\,\mathrm{KI}} = \frac{\mathrm{AC}^2}{\mathrm{FK}^2}; \text{ or } \frac{\frac{1}{2}\,\mathrm{AB}\,.\,\mathrm{CD}}{\frac{1}{2}\,\mathrm{FG}\,.\,\mathrm{KI}} = \frac{\mathrm{AC}^2}{\mathrm{FK}^2};$$

is, $\dfrac{\text{Area ACB}}{\text{Area FGK}} = \dfrac{\mathrm{AC}^2}{\mathrm{FK}^2}$; and this expressed in the form of a ortion, becomes

$$\text{Area ACB} : \text{Area FGK} :: \mathrm{AC}^2 : \mathrm{FK}^2.$$

general it may be shown that the areas of all similar es are to each other as the squares of their like sides.*

ercise. From a given triangle, ABC, (see fig. Art. let it be required to cut off a part, $r\,m\,c$, equal to oneh of the area ABC, by a line $r\,m$ parallel to the side AB.

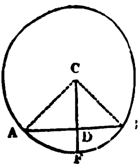
$$\text{Here, Area ABC} : \text{Area } rmc :: \mathrm{CB}^2 : cm^2;$$
$$\text{or } 1 : \tfrac{1}{4} :: \mathrm{CB}^2 : cm^2;$$
$$\therefore\ cm^2 = \frac{\mathrm{CB}^2}{4}, \text{ and } cm = \frac{\mathrm{CB}}{2}.$$

HEOREMS AND PROBLEMS RELATIVE TO THE CIRCLE.

1. **THEOREM.** A straight line CD, drawn from the

Def. Similar figures have their several angles equal, each to each, he *sides about the* equal angles proportional.

centre C of a circle, to the middle point D of a chord AB, is perpendicular to that chord.



The triangles ODB and ODA are exactly alike, for $CB = CA$, $DB = DA$, and CD belongs to them both; therefore $\angle CDB = \angle CDA$, that is, the line CD is perpendicular to AB.

If CBE be folded over on the line CDE, the point B will fall upon A, and the arc BE will cover the arc EA. Hence it follows that the line which bisects the chord also bisects the arc.

The converse of this theorem is also true; that is, if CD be drawn at right angles to AB, it will bisect it, and moreover the arc AEB will be bisected in the point E.

Because $OB = OA$, the triangle ABO will be isosceles, and therefore $\angle B = \angle A$; then in the triangles OBD and OAD, we have OD common to both; the angles at D equal, being right angles; $\angle B = \angle A$; and therefore (Art. **34.**) the remaining $\angle BOD$ will be equal to the remaining $\angle AOD$; hence it follows (Art. **16.**) that the triangles are exactly alike, and therefore $DB = AD$.

Cor. A line DC bisecting any chord AB at right angles passes through the centre C of the circle.

Exercises.

1. The radius AC of a circle is 10, and the chord AB is 16, required the perpendicular CD.

Here $AD = \frac{1}{2} AB = \frac{1}{2}$ of $16 = 8$; then from the right-angled triangle ADC, we have,

$$CD^2 = AC^2 - AD^2 = 10^2 - 8^2 = 36; \therefore CD = 6.$$

2. Required CD, when $AC = 20$, and $AB = 24$. *Ans.* 16.
3. Required AB, when $AC = 15$, and $CD = 9$. *Ans.* 24.
4. The radius $AC = 5$, and the versed sine $DE = 2$, required the chord AB.

$$\text{Here } CD = CE - DE = 5 - 2 = 3; \text{ then,}$$

$$AD = \sqrt{5^2 - 3^2} = 4, \text{ and } \therefore AB = 2\,AD = 2 \times 4 = 8.$$

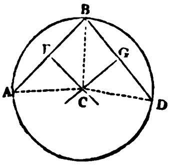
5. *Required* AB when $AC = 8$, and $DE = 3$. *Ans.* 12·49.
6. *Required* AC, when $AB = 64$, and $DE = 16$.

Let $x = AC$; then $AD = \frac{1}{2}$ of $64 = 32$; $CD = CE - DE = x - 16$; and

$\therefore AC^2 = AD^2 + DC^2$, or $x^2 = 32^2 + (x - 16)^2$; solving this equation we find $x = 40$.

7. Required AC, when $AB = 8$ feet, and $DE = 2$ feet.
Ans. 5 *feet.*

57. Problem. To find the centre of a given circle ABD.



Draw any two chords AB and BD; bisect these chords by the perpendiculars FC and GC, intersecting each other in the point C; then this point will be the centre of the circle.

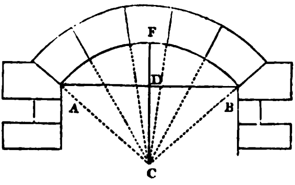
Because, by the last theorem, the centre of the circle will lie in the line FC; and for the same reason, it will also lie in the line GC; therefore it must be in the intersection C.

58. Problem. To draw a circle through three given points, A, B, and D. (See the last figure.)

Join two of these points by the straight lines AB and BD; bisect these lines by the perpendiculars FC and GC, and the intersection C will be the centre of the circle. Then on C as a centre, with the radius CA, describe the circle ABD, passing through the given points.

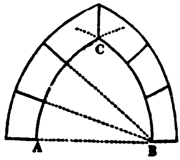
Application of this problem.

1. To construct an arch. Let AB be the span of an arch, and DF its perpendicular height. Find, by the preceding problem, the centre C of the circle to pass through the given points A, F, and B. Divide the arc AFB into any convenient number of equal parts; join these points of division and the centre C, and it will *give the joints of the* arch-stones. The arch-stones, being

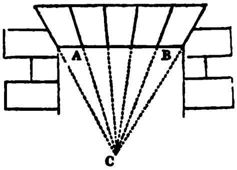


in the form of a wedge, mutually prevent each other from falling.

2. To construct a gothic arch. Let A B be the span of the arch. Upon A as a centre, and with the radius A B, describe the circle B C; and then upon B as a centre, with the same radius, describe the circle A C, intersecting the former in the point C; divide the arcs A C and B C into any convenient number of equal parts; join these points of division in the arc A C with the centre B, and those in the arc B C with the centre A, and it will give the joints of the arch-stones.

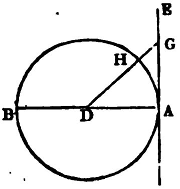


3. To construct an arch when the intrados A B is a straight line. Divide the span A B into any convenient number of equal parts; and upon A B construct the equilateral triangle A B C; then C in this case will be the centre to which the lines, forming the joints of the stones, must be drawn.



59. THEOREM. A line A E perpendicular to the extremity of the radius D A is a tangent to the circle.

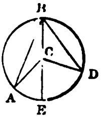
For take any point G in the line A E, and join G and the centre of the circle D; then, since D A G is a right-angled triangle, the hypotenuse D G (*Cor.* 1. Art. **45.**) will be greater than D A or D H; and consequently the point G must lie without the circle; and as the same thing may be shown for every other point in the line A E, it follows that this line only meets the circle in the point A, or, in other words, A E is a tangent to the circle.



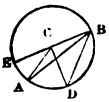
The converse of this theorem is true, that is, *if* A E *be a tangent to the circle, the radius* D A *will be perpendicular to* *A E.*

For since the tangent AB lies entirely without the circle except at the point of contact A, the point G will be without the circle, and any line DG will be greater than the radius DH or DA; in other words, DA will be the shortest line that can be drawn from D, to meet the line AB; and therefore (Art. **25.**) DA is perpendicular to AB.

60. THEOREM. An angle ACD at the centre C of a circle is double the angle ABD at the circumference, upon the same arc AED.



There are two cases in this theorem: first, when the centre C lies within the angle ABD, and second, when the centre C lies without the angle ABD.



Join BC, and produce it to E. The triangle ACB is isosceles; and $\therefore \angle CBA = \angle CAB$; but by Art. **34.** the external $\angle ACE = \angle CBA + \angle CAB$;

$$\therefore \angle ACE = 2\angle CBA.$$

In precisely the same way we have,

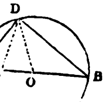
$$\angle DCE = 2\angle CBD.$$

Then, in order to prove the theorem in the first case, we have, by adding these equals,

$\angle ACE + \angle DCE = 2\angle CBA + 2\angle CBD$; $\therefore \angle ACD = 2\angle ABD$.

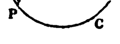
To prove the theorem in the second case we have, by subtracting,

$\angle DCE - \angle ACE = 2\angle CBD - 2\angle CBA$; $\therefore \angle ACD = 2\angle ABD$.



Cor. 1. All angles BAD, BED, &c., standing on the same arc, are equal, and they are measured by half the arc BD: because each of these angles is equal to half the angle BCD.

Cor. 2. The angle ADB in the semicircle ADB is a right angle; because it is measured by half the arc ACB, or $\frac{1}{2}$ of $180° = 90°$.



Cor. 3. If AQ be a tangent to the circle at the point A, then the $\angle$ DAQ, cut off by the chord AD, is equal to the $\angle$ B, at the circumference of the circle, standing on the arc AD cut off.

For since AB is a diameter, $\angle QAB =$ a right angle; but $\angle DAB + \angle B =$ a right angle;

$$\therefore \angle QAB = \angle DAB + \angle B;$$

from these equals take away $\angle DAB$;

$$\therefore \angle DAQ = \angle B.$$

Applications of the preceding theorems.

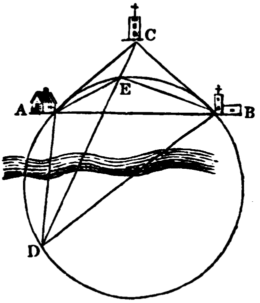
1. On a given line AD, to describe a circle ADBC, which shall contain the angle B equal to a given angle. (See the last figure.)

Draw AQ, making the angle DAQ equal to the given angle (Art. **26.**). Draw AB perpendicular to AQ; from D draw DO, making the angle ADO equal to DAB; then O will be the centre of the circle required. (*Cor.* 3. Art. **60.**)

2. The distance between two objects A and D is 3 miles. (See the preceding figure.) A surveyor takes the angle APD (=45°) which the two objects form at the station P; he then measures PD (= 4 miles). Required to find by construction his distance from the object A.

Ans. 1·8 *or* 3·8 *miles.*

From a scale of equal parts take off AD = 3; and by the last exercise describe a circle ACBD to contain the angle APD = 45°; then on D as a centre, with a radius containing 4 parts of the scale, describe an arc cutting the circle ACD in the points P and B; join PA, and it will be the distance required. It is obvious that the arc described upon D as a centre will cut the circle ACD in two points P and B; hence the question admits of two answers; that is, the units in PA or BA will satisfy the conditions of the question.*



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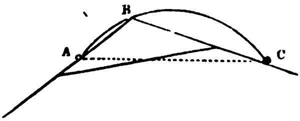
* *This problem* admits of another method of construction.

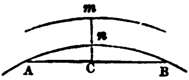
The distances of three objects A, B, and C, from each er are as follows: AB = 12 miles, BC = 7·2 miles, and AC 8 miles. From a station D, I took the angles BDC = 25°, d CDA = 19°. Required my distance from the object C.

Construct the triangle ABC; from B draw BE, making the ABE = 19°, that is the same as ∠ CDA; in like manner om A draw AE, making the ∠ BAE = 25°, that is the same s ∠ BDC; through the three points A, B, and E (Art. **58.**) lescribe the circle ADBE; join CE and produce it until it intersect the circle in the point D, then this point will be the station, and the units in DC will be the distance required = 15 miles. Because by Art. **60.**, ∠ ABE = ∠ ADE, and ∠ BAE = ∠ BDE.

4. To describe the arc of a circle through three given points A, B, and C, without finding the centre. Fix two pins at A and C; take two laths, AB and CB, and fix them in the manner shown in the figure. Move the triangular frame, thus formed, round, constantly keeping the sides BA and BC pressing against the pins A and C; then a pencil applied at the vertex B will trace the circle required. This method is frequently employed by workmen when the radius of the circle is very great.

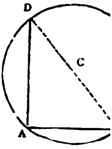


5. To find the joint nm of an arch AnB, without knowing the centre of the circle. On each side of the given point n take nA = nB; draw AB and bisect it with the perpendicular Cnm (Art. **22.**); then nm will be the joint of the arch-stone required. For mC will evidently run to the centre of the circle.

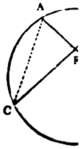


6. To draw a perpendicular AD to a given line AB from the extremity A.

On any point C as a centre, with CA as a radius, d the circle ADG, cutting the line AB in G; join GC, and produce it until it cuts the circle in the point D; join DA, and it will be the perpendicular required. Because DAG will be a semicircle, and therefore the $\angle A$ will be a right angle.

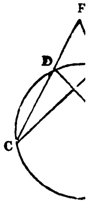


61. THEOREM. If two straight lines cut each other, in the point F, either within or without the circle, the rectangle of the parts of the one is equal to the rectangle of the parts of the other, that is, $FB \cdot FA = FC \cdot FD$.



Join AC and DB; then, in both figures, the triangles FDB and FAC are similar; because, *Cor.* 1. Art. **60.**, $\angle C = \angle B$; $\angle F$ of the one equals $\angle F$ of the other; and therefore the remaining $\angle FDB$ of the one will be equal to the remaining $\angle FAC$ of the other; hence we have, by Art. **54.**,

$$FB : FC :: FD : FA; \therefore FB \cdot FA = FC \cdot FD.$$



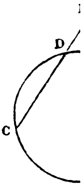
Cor. 1. When CAD, in the first figure, is a semicir AF is taken perpendicular to CD; then $FA = FB$, and

$$\therefore FA^2 = FC \cdot FD,$$

that is, the square of the ordinate in a semicircle is the product of the abscissa.

Cor. 2. Let the line FB, in the second figure, on the point F until it comes to touch the circle, as in the annexed figure; then the part AB will vanish, and $FB \cdot FA$ will become FA^2; in this case, therefore, we have,

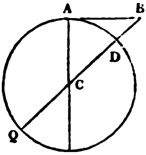
$$FA^2 = FC \cdot FD;$$



that is, the square of the line touching the circle is equal to the product, or rectangle, of th *secant* into the part without the circle.

Applications of the preceding Properties.

1. To determine the correction in levelling, arising from the earth's curvature.



Let AB be the direction of the level line taken from A, then AB will be a tangent to the earth's surface at A; and therefore if C be the centre of the earth, AB will be perpendicular to AC. Draw BQ through the centre C; then in the distance AD, the deviation of the *apparent* from the *true* level will be the line BD; but by *Cor.* 2., we have,

$$BQ . BD = AB^2; \therefore BD = \frac{AB^2}{BQ}.$$

But as BD is, in all actual surveys, very small as compared with the diameter of the earth, the line BQ may be taken equal to DQ, without incurring any sensible error; and for the same reason AB may be regarded as the same as AD. Let $d = DQ$, the diameter of the earth (7960 miles nearly), then,

$$BD = \frac{AB^2}{d}.$$

If AB be taken one mile, then $BD = \frac{1}{7960}$ miles, = 8 inches nearly. Hence for every mile of survey the true level, or surface of the earth, is 8 inches below the apparent level.

2. To find the distance at which an object may be seen at sea.

Let the height of the object $BD = h$; and the distance $AB = c$; then we have as before,

$$d . BD = AB^2, \text{ that is, } dh = c^2; \therefore c = \sqrt{dh}.$$

Ex. 1. The Peak of Teneriffe is about $2\frac{1}{2}$ miles above the level of the *sea; at what* distance can it be seen?

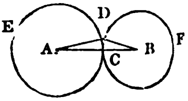
Here, $h = 2\frac{1}{2}$; $\therefore\ c = \sqrt{7960 \times 2\frac{1}{2}} = 141$ miles.

Ex. 2. The top of a tower was seen at the distance of 25 miles, required its height. *Ans.* 414 *feet.*

Ex. 3. At what distance may the Peak of Teneriffe be seen from the top-mast of a ship 80 feet above the surface of the sea? *Ans.* 152·04 *miles.*

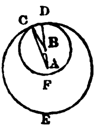
Ex. 4. If a mountain 1 mile above the level of the sea can be seen at the distance of 89 miles, what must be the diameter of the earth? *Ans.* 7921 *miles.*

62. THEOREM. When the distance AB, between the centres A and B of two circles, is equal to the sum of their radii AC and CB, the circles will touch each other externally.



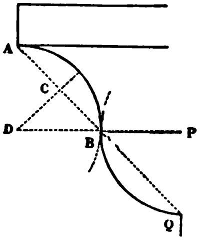
The circles will obviously pass through the point C, but they will have no other point in common; for let D be any point in the circumference of the circle A; join AD and BD; then, in the triangle ADB, the sides AD+DB will be greater than the side AB; from this inequality take away AD or AC, then DB will be greater than CB, that is, the point D must lie without the circle B; and the same may be shown for any other point in the circle A. Therefore the circles will only meet each other in the point C; that is, they will touch each other in this point.

63. THEOREM. When the distance AB, between the centres A and B of two circles, is equal to the difference of the radii AD and BD, the circles will touch each other internally.

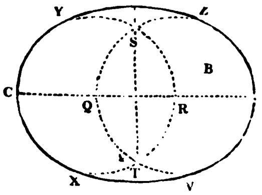


Proceeding as in the last theorem, let C be a point in the circumference of the small circle DF; join BC and AC; then, in the triangle ABC, the side AC will be less than the sum of the sides AB+BC; but since BC = BD, therefore AB+BC = AD; and consequently AC will be less than AD, that is, the point C lies within the large circle DE; and the same may be shown for any other point in the small circle DF. Therefore, &c.

Applications of the two preceding Theorems.

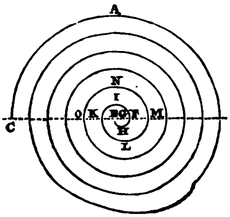


1. To draw a *Cima recta* ABQ. Join AQ, and bisect it in the point B; bisect AB by the perpendicular DC; take at pleasure any point D in this line as a centre, and with the radius DB describe the arc AB. Join DB, and produce it until BP = BD; then on P as a centre with the radius PB describe the arc BQ. It is obvious from Art. 62., that the circles touch each other in the point B, and therefore the serpentine curve ABQ forms a continuous line.



2. To describe an oval CVDY. Divide the major diameter CD into three equal parts in the points Q and R. On Q and R as centres describe the circles CYX and DZV, intersecting each other in the points S and T. Join TQ, and produce it until it intersects the circle CYX in the point Y; then on T as a centre, with the radius TY, describe the arc YZ; and in like manner on S as a centre describe the arc XV. The arcs here described will obviously touch each other in the points Y, Z, V, and X.

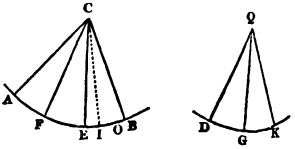
By dividing CD into four equal parts a more elongated oval may be formed.



3. To describe a spiral CAL. Take G for the eye of the spiral, and EG = GF as the radius of the smallest semicircle EHF. On G as a centre, with the given radius GE, describe the semicircle EHF; then on E as a centre, with the radius EF, describe the semicircle FIK; now on *G as a centre, with* the radius GK,

describe the semicircle KLM; and so on to any extent, observing to take the centres G and E alternately.

64. THEOREM. In the same or equal circles the angles ACB and DQK at the centre are in the ratio of their arcs AB and DK.



Let, for example, the angles ACB and DQK be to each other as 3 to 2, that is, let the lines CF, CE divide the ∠ ACB into three equal parts, and QG the ∠ DQK into two equal parts; then ∠ ACF = ∠ DQG, and the sector ACF will be identical with the sector DQG, and also the arc AF = the arc DG, and so on (Art. **13.**); consequently the ∠ ACB will have the same ratio to the ∠ DQK that the arc AB has to the arc DK; that is,

$$\frac{\angle \text{ACB}}{\angle \text{DQK}} = \frac{\text{arc AB}}{\text{arc DK}}, \text{ or in the form of a proportion,}$$

$$\angle \text{ACB} : \angle \text{DQK} :: \text{arc AB} : \text{arc DK}.^{*}$$

65. PROBLEM. To inscribe a square in a given circle.

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* If the angles are incommensurable. Let the less angle DQK be placed upon the greater, so that ∠ ACE = ∠ DQK, and arc AE = arc DK; then if the theorem above demonstrated is not true, let

∠ ACB : ∠ DQK :: arc AB : arc AO.

Conceive the arc AB to be divided into any number of equal parts, each of which is less than EO, then there will be at least one point I of the division between E and O. Then by the above theorem, we have,

∠ ACB : ∠ ACI :: arc AB : arc AI.

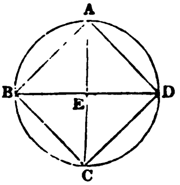
Now in these two proportions the antecedents are the same;

∴ ∠ DQK : ∠ ACI :: arc AO : arc AI.

Here arc AO is greater than AI, and if therefore this proportion be true, ∠ DQK must be greater than ∠ ACI; but this is not the case, on the contrary it is less; hence it follows, that it is absurd to suppose that ∠ ACB : ∠ DQK :: arc AB : an arc AO *greater* than AE.

In the same manner it may be shown that the last term of the proportion cannot be *less* than AE; hence it follows that AE itself must be the *fourth term, that is,*

∠ ACB : ∠ DQK :: arc AB : arc AE or DK.



Draw two diameters BD and AC at right angles to each other; join the points A, B, C, D; and then ABCD will be the required square. For in the triangles ABE and ADE, we have BE = ED, AE common to both, and the ∠BEA = ∠DEA, being both right angles; therefore the triangles are identical; and AB = AD; in the same manner it may be shown, that AD = DC = CB. Moreover, because BAD is a semicircle (Art. **60.**), the ∠BAD is a right angle; hence ABCD is a square.

An octagon may be inscribed, by bisecting the arcs AD, DC, &c., and joining these points of division.

Exercises.

1. Let r = the radius of the circle; required the side of the inscribed square.

From the right-angled triangle AEB, we have,

$$AB^2 = BE^2 + AE^2 = 2r^2; \therefore AB = r\sqrt{2}.$$

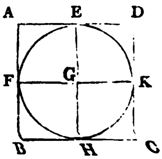
2. The radius of the circle = 4; required the area of the inscribed square. *Ans.* 32.

3. The area of the inscribed square = 18; required the diameter of the circle. *Ans.* 6.

66. PROBLEM. To describe a circle about a given square.

Let ABCD be the given square (see the last fig.); draw the diagonals AC and BD, cutting each other in the point E; then these diagonals will bisect each other, making EA = ED = EC = EB (Why?). Therefore E will be the centre of the circumscribed circle required.

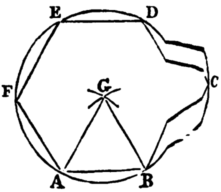
67. PROBLEM. To describe a square about a given circle.



Draw two diameters, FK and EH, at right angles to each other; and through the points F, H, K, and E, draw AB, BC, CD, and DA, touching the circle; then ABCD will be the square required.

Let it also be required, to inscribe a circle FHKE *in a given square* ABCD.

68. Problem. To inscribe a regular hexagon in a given circle.



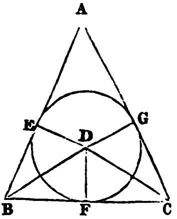
Suppose the problem done, and that ABCD, &c., is the inscribed hexagon; draw the radii AG and BG. Now the $\angle AGB = \frac{1}{6}$ of $360° = 60°$; and because $GA = GB$, the $\angle GAB = \angle GBA$; and as the sum of all the angles of the triangle is equal to 180°, it will be readily seen that $\angle GAB$ or $\angle GBA = 60°$; therefore the triangle AGB is equilateral, and hence the side of the inscribed hexagon is equal to the radius of the circle.

Cor. 1. The chord of 60° is equal to the radius.

Cor. 2. By joining the points A, C, and E, we should obviously form the inscribed equilateral triangle.

Cor. 3. ***Regular polygons*** are not only equilateral, but also equiangular; thus, $\angle ABC = \angle BCD =$ &c.

69. Problem. To inscribe a circle in a given triangle.



Let ABC be the given triangle; bisect the angles BCA and CBA (Art. **21.**) by the lines CD and BD meeting each other in the point D; from D let fall DF perpendicular to BC, then DF will be the radius of the inscribed circle.

Draw DE and DG perpendicular to BA and CA; then the triangles BDF and BDE will be identical (Why?); and therefore $DE = DF$. In like manner it may be shown that $DG = DF$. Hence the circle described on the centre D, with the radius DF, will pass through the points F, E, and G; and by Art. **59.**, the circle will likewise touch the sides of the triangle in these points.

Exercises.

1. Let $r = DF$, the radius of the inscribed circle, the side $BC = a$, $BA = c$, and $AC = b$; required an expression for the area of the triangle.

Here the triangle ABC is made up of three triangles, viz. BCD, DA, *and* BAD.

$$\text{Area BCD} = a \times \frac{r}{2}; \quad \text{Area CDA} = b \times \frac{r}{2}; \quad \text{Area BAD} = c \times \frac{r}{2};$$

$$\therefore \text{Area ABC} = a \times \frac{r}{2} + b \times \frac{r}{2} + c \times \frac{r}{2} = (a+b+c)\,\frac{r}{2}.$$

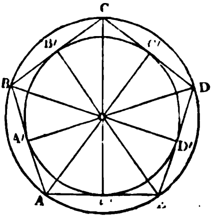
2. The two perpendicular sides of a right-angled triangle are 8 and 6; required the radius of the inscribed circle.

Ans. 2.

3. The base of a right-angled triangle is 18, and the hypotenuse is 30; required the radius of the inscribed circle.

Ans. 6.

70. Theorem. A circle may be described about, or inscribed within, any regular polygon, and conversely.

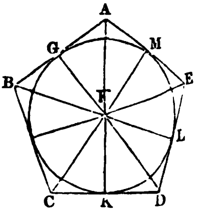


Let ABCD, &c., be a regular polygon; through the three points B, C, and D, (Art. 58.) describe a circle, of which the centre is O, B′ and C′ being the middle of the chords BC and CD. Join OB and OE; then if the quadrilateral OC′DE be folded over upon OC′, it will exactly cover the quadrilateral OC′CB (Why?); and therefore OE = OB, that is, the circle will also pass through the point E of the polygon. In precisely the same way it may be shown that the circle which passes through the points C, D, and E, will also pass through the next point A of the polygon, and so on.

Again, as BC, CD, DE, &c., are all equal chords, the perpendiculars OB′, OC′, OD′, &c., will be all equal to each other, and therefore the circle described on O as a centre, with the radius OB′, will touch the chords in the points B′, C′, D′, &c., that is, the circle will be inscribed in the polygon.

As the angles BOC, COD, &c., are all equal to each other, each of these angles will be a certain part of 360° indicated by the number of sides in the polygon. To inscribe a polygon, therefore, in a given circle, we must divide the circumference into as many equal parts as there are sides in the polygon, and then these points being joined will give the polygon required. And in the case of the circumscribed *polygon, we must* draw tangents to these points of division.

71. Theorem. The area of a regular polygon is equal to the rectangle of the perimeter (or sum of the sides) by half the radius of the inscribed circle.

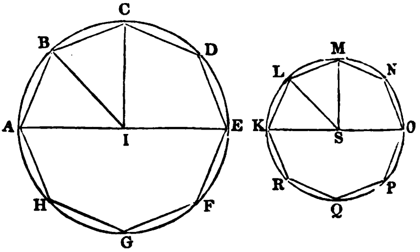


Let FG be the radius of the inscribed circle, then,
the area of the triangle $FBA = AB \times \frac{1}{2} FG$.

Now if there are n sides in the polygon, there will be n triangles identical with FBA,

$\therefore$ Area polygon $= n \times AB \times \frac{1}{2} FG$;
but $n \times AB =$ the perimeter,
$\therefore$ Area polygon $=$ the perimeter $\times \frac{1}{2} FG$.

72. Theorem. The perimeters of two regular polygons of the same number of sides are to each other as the radii of the circumscribed circles; and moreover their areas are to each other as the squares of the radii.



Let I and S be the centres of the circumscribed circles; then the triangles IBC and SLM are equiangular, and

$$\therefore BC : LM :: IB : SL.$$

If n be the number of sides in each polygon, then multiplying the first and second terms of this proportion by n, we have,

$$n \times BC : n \times LM :: IB : SL;$$

that is, the perimeter of the polygon ABCD, &c. is to the perimeter of the polygon KLMN, &c. as IB is to SL. In like manner it may be shown that the perimeters are to each other as the radii of the inscribed circles.

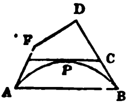
Again, by Art. **55.**, we have,

$$\text{Area } ABI : \text{area } KLS :: AI^2 : KS^2;$$

$$\text{or, } n \times \text{area } ABI : n \times \text{area } KLS :: AI^2 : KS^2;$$

that is, the area polygon ABCD, &c. is to area polygon KLMN, &c. *as* AI^2 *is to* KS^2.

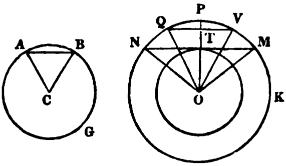
73. Theorem. The chord AB is less than the convex curve APB; and the line ADB is longer than the convex curve which it envelopes.



First, as AB is a straight line, it must be shorter than any other line APB which can be drawn between the points A and B.

Again, if APB is not shorter than all the lines which enclose it; among these, some line ADB must be shorter than all the others. Draw the straight line FC touching the curve in the point P and cutting AD and DB in F and C; then as FC is shorter than FDC, it follows that AFCB must be shorter than AFDCB; but by hypothesis, AFDCB is the shortest line, which is absurd; hence the hypothesis is false, and therefore the enveloping lines must be longer than the convex curve APB.

74. Theorem. The circumferences of circles are to each other as their radii. That is, cir. circle CA : cir. circle OQ :: CA : OQ.



If this theorem is not true, CA : OQ :: cir. circle CA : cir. of a circle greater or less than OQ. First suppose it less, and if possible let CA : OQ :: cir. circle CA : cir. circle OT.

Draw NM, touching the circle OT in the point T; produce OT to P, and let PK be a quadrant of this circle. Take away the half of PK, and then the half of the remainder, and so on, until we have an arc PV less than PM; draw VQ perpendicular to OP; then VQ will obviously be a side of a regular inscribed polygon: let AB be the side of the polygon, inscribed in the circle CA, having the same number of sides: then, Art. 72.,

per. polygon AB : per. polygon QV :: CA : OQ.

But by hypothesis,

CA : OQ :: cir. circle CA : cir. circle OT;

∴ per. poly. AB : per. poly. QV :: cir. circle CA : cir. circle OT.

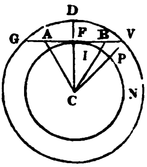
Now, this proportion is false, for (Art. 73.) the per. polygon AB is less than the cir. circle CA, whereas the per. polygon QV is greater than the cir. circle OT; therefore it is impossible that CA can be to OQ as *cir. circle* CA is to a circumference less than the *cir. circle* OQ.

In precisely the same way it may be shown, that it is impossible that OQ can be to CA as the cir. circle OQ is to a circumference less than the cir. circle CA; or, what amounts to the same thing, CA cannot be to OQ as cir. circle CA is to the cir. circle greater than OQ.

Having shown that the fourth term of our proportion can neither be less nor greater than cir. circle OQ, it follows that,

$$\text{OA} : \text{OQ} :: \text{cir. circle OA} : \text{cir. circle OQ}.$$

75. Theorem. The area of a circle is equal to the rectangle contained by its circumference and half the radius: that is, area circle CD = cir. circle CD × $\frac{1}{2}$ CD.



For if cir. circle OD × $\frac{1}{2}$ CD is not equal to the area of the circle whose radius is CD, it must be equal to the area of a circle either greater or less. Let us suppose it to be equal to the area of a less circle whose radius is OF; that is, if possible, let cir. circle CD × $\frac{1}{2}$ CD = area circle CF.

Draw GV, a tangent to the point F; and join CV, cutting the circle CF in P. Let FN be a quadrant of the circle CF; take away the half of FN, and then the half of the remainder, and so on, until we have an arc FI less than the arc FP; take FA = FB, then AB will be the side of a regular polygon circumscribed about the circle CF. By Art. 71., the area of this polygon = its perimeter × $\frac{1}{2}$ CF. But this perimeter is less than the cir. circle CD, and also OF is less than CD; therefore the area of the polygon is less than cir. circle OD × $\frac{1}{2}$ OD, which by supposition is equal to the area of the circle OF; that is, the area of the polygon is less than the area of the circle OF; but it is manifestly greater, since the circle lies wholly within the polygon; hence it is impossible that cir. circle CD × $\frac{1}{2}$ OD can be equal to a circle less than the circle OD: and by a similar construction about the circle GDV, it may be shown that it is not greater. Therefore, &c.

Cor. Let π = the circumference of a circle whose diameter is unity, then as the circumferences of circles are to each other as their radii or diameters, we shall have, (see figure first to Art. 74.)

$$\pi : \text{circum. ABG} :: 1 : 2\,\text{AC};$$

$$\therefore \text{circum. ABG} = 2\,\pi \times \text{AC};$$

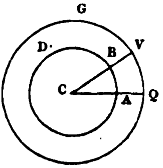
and then by the foregoing theorem,

$$\text{area } ABG = \text{circum. } ABG \times \tfrac{1}{2} AC$$
$$= 2\pi \times AC \times \tfrac{1}{2} AC = \pi \times AC^2.$$

Here as π is a constant quantity, it follows that the areas f circles are as the squares of their radii.

In order to find the area of a given circle we must be able determine the numerical value of π; this will be done in future article.

76. The two preceding theorems may be proved more asily in the following manner:—



Let ABD and QVG be any two circles, aving the common centre C. Suppose the rcumference of ABD to be divided into any umber of indefinitely small parts AB; oduce CB to V, and CA to Q; then if AB contained in the circumference ABD n mes, QV will be contained n times in QVG. s AB is very small (and it may be taken small as we please), it may be regarded as a straight line, and en ACB and QCV will be similar triangles, and hence we have

$$AB : QV :: CA : CQ;$$
$$\text{or, } n \times AB : n \times QV :: CA : CQ;$$

t $n \times AB$ = cir. circle CA; and $n \times QV$ = cir. circle CQ;

$$\therefore \text{cir. circle } CA : \text{cir. circle } CQ :: CA : CQ,$$

hich establishes the theorem Art. **74**.

Again, area $CAB = AB \times \tfrac{1}{2} CA$,

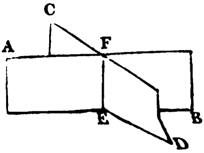
$$\therefore n \times \text{area } CAB = n \times AB \times \tfrac{1}{2} CA,$$

that is, area circle CA = cir. circle CA × $\tfrac{1}{2}$ CA,

hich establishes the theorem, Art. **75**.

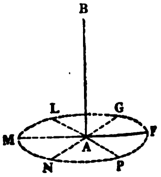
HEOREMS, ETC. RELATIVE TO PLANES AND SOLIDS.

77. DEFINITIONS.

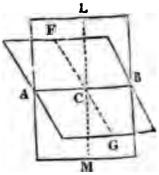


1. If AB and CD be two planes cutting ch other in EF, this line is called the mmon section.

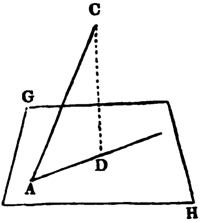
2. *A straight line is* perpendicular to



a plane when it is perpendicular to every line meeting it in that plane. Thus let BA be perpendicular to the plane LMN, &c., and from the foot A of the perpendicular, let any straight lines AL, AM, &c. be drawn in the plane; then the angles BAL, BAM, &c. will all be right angles.



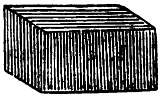
3. Let AB be the common section of two planes; and let FG and LM be drawn, in these two planes, at right angles to AB; then the angle LCF is the measure of the *inclination* of the two planes. If the angle LCF is a right angle, then the plane ALBM is said to be perpendicular to the plane AFBG.



4. Let GH be a plane, and AC a straight line inclined to it; from C let fall CD perpendicular to the plane, and join AD; then the angle CAD is the inclination of AC to the plane GH.

5. Parallel planes are such as being produced ever so far in every direction will never meet.

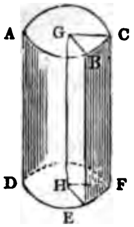
6. A *prism* is a solid whose ends are parallel, and identical plane figures; and the sides or faces which connect the ends are parallelograms. A prism is named according to the figure of its base: thus, a triangular prism is one that has a triangular base, a rectangular prism is one that has a rectangular base, and so on.



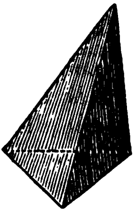
7. A *parallelopiped* is a prism bounded by six parallelograms, every opposite two of which is parallel and identical. A rectangular parallelopiped has all its faces rectangles. A *cube* is a square prism, having six equal square faces.

8. A *cylinder* is a prism having circles for its ends. It is

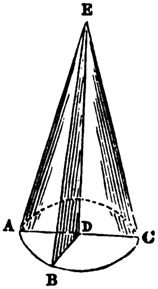
formed by the revolution of a straight line BE about two equal and parallel circles ABC and DEF, and this revolving line is always parallel to the axis GH, which joins the centres of the two circles.



9. A *pyramid* is a solid, having any right-lined figure for its base, and all its side faces plane triangles, whose vertices meet in a point called the vertex of the pyramid.



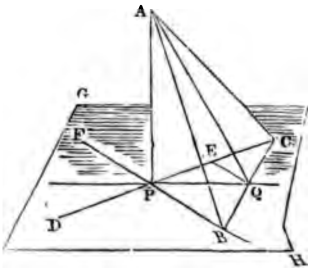
10. A *cone* is a pyramid having a circular base. It is formed by the revolution of the line EB about the fixed point E, the other extremity B being constantly kept in the circumference ABC of the base. The line ED, joining the vertex and the centre of the base, is called the *axis* of the cone. In a *right cone* this axis is perpendicular to the base.



11. A *sphere* is a solid bounded by a curve surface, which is everywhere at the same distance from a certain point called the centre. A sphere is formed by the revolution of a semicircle about its diameter. This diameter is in consequence called the axis of the sphere.

12. The *altitude* of a solid is the perpendicular drawn from the vertex to the base.

78. THEOREM. If a straight line AP is perpendicular to two other straight lines PC and PB, cutting each other at its foot in the plane GH, it will also be perpendicular to the plane GH.



Through P draw any straight line PQ in the plane GH; draw BO, through any point Q in the *straight line* PQ, *making* BQ =

QC; then from the triangles BCP and BCA, we have by Art. 51,

$$PB^2 + PC^2 = 2\,PQ^2 + 2\,QC^2;\quad AB^2 + AC^2 = 2\,AQ^2 + 2\,QC^2;$$

subtracting the first equality from the second,

$$AB^2 - PB^2 + AC^2 - PC^2 = 2\,AQ^2 - 2\,PQ^2;$$

but from the right-angled triangles APB and APC, we have,

$$AB^2 - PB^2 = AP^2, \text{ and } AC^2 - PC^2 = AP^2;$$

$$\therefore\ 2\,AP^2 = 2\,AQ^2 - 2\,PQ^2,$$

and $\therefore\ AP^2 = AQ^2 - PQ^2$, or $AQ^2 = AP^2 + PQ^2$. Hence APQ is a right-angled triangle, and AP is perpendicular to PQ.

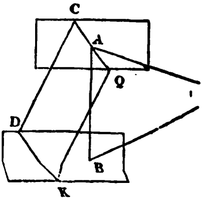
Cor. 1. It is evident that there can be only one perpendicular drawn from a given point A to meet the plane GH, and that this perpendicular is the shortest line.

Cor. 2. Let AP be perpendicular to the plane GH, and BC any line in that plane; if from the foot P of the perpendicular, the line PQ be drawn perpendicular to BC, then QA drawn to any point in the line AP, will be perpendicular to BC.

Take BQ = QC, and join CP, BP, BA, CA; then the triangles BPQ and QPC will be identical (Art. **16.**), and therefore PB = PC. The triangles BPA and CPA will also be identical, and therefore AB = AC. The triangles ABQ and ACQ will be identical (Art. **19.**), and therefore ∠ AQB = ∠ AQC, that is, AQ will be perpendicular to BC. It is evident that BC is perpendicular to the plane APQ.

Cor. 3. If AP be at right angles to each of the three straight lines PB, PQ, and PC, then these three straight lines will be in the same plane.

79. THEOREM. Two planes A and B perpendicular to the same straight line AB, are parallel to each other.



If possible, let the planes meet in a point P; join AP and BP; then, because the line AB is perpendicular to the two planes A and B, the angles PAB and PBA are right angles, and therefore AP is parallel to BP, which is contrary to the supposition; *therefore* the planes cannot meet *in P, and must* therefore be parallel.

If A and B be two parallel planes, and AB is perpen- to one of them, it will also be perpendicular to the

THEOREM. If two parallel planes A and B be cut by plane CQKD, the intersections CQ and DK are paral- ee the last figure.)

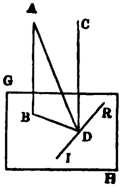
s the parallel planes cannot meet, neither can the lines DK drawn in these planes. But CQ and DK are in the ne, therefore they are parallel to each other.

The parallel lines OD and QK, between two parallel planes , are equal.

HEOREM. If AB is perpendicular to the plane GH, any parallel to AB will be perpendicular to the same plane.

plane pass through the parallel lines AB cutting the plane GH in the line BD; in e GH draw IR perpendicular to BD, and

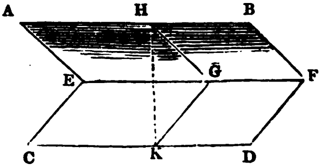
or. 2, Art. 78., IR is perpendicular to e ABD; therefore ∠ IDC is a right angle; ∠ CDB is also a right angle, because AB ndicular to BD, and CD is parallel to AB. CD is perpendicular to the two lines DI it is therefore perpendicular to the plane GH.



1. Conversely, if AB and CD are perpendicular to e GH, they will be parallel; for the line CD, which lel to AB, is by this theorem perpendicular to the H, and there can be only one line drawn from the perpendicular to the plane.

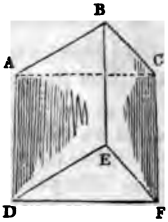
HEOREM. Two lines AB and CD parallel to a third are parallel to each other.

HK be a plane perpen- tó EF; then the lines CK, being parallel to by the preceding the- perpendicular to the IK, and therefore, by they will be parallel to er.



83. THEOREM. If two angles ABC and DEF, have the sides BA parallel to ED, and BC to EF, then those angles will be equal.

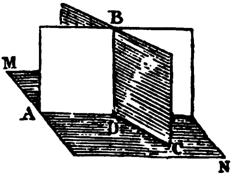
Take BA = ED, BC = EF, and join the different points in the figure.



The figure ABED is a parallelogram (Art. **36.**), and therefore AD = BE; for a similar reason BCFE is a parallelogram, and therefore CF = BE.

Now because AD and CF are both parallel and equal to BE, it follows, Art. **82.**, that AD is parallel and equal to CF; therefore ACFD is a parallelogram, and therefore DF = AC. Hence the triangles ABC and DEF are identical, and $\angle$ ABC = $\angle$ DEF.

84. THEOREM. If the straight line BD is perpendicular to the plane MN, any plane ABD passing through BD will be perpendicular to the plane MN.



Let AD be the intersection of the plane AB with the plane MN; in the plane MN draw DC perpendicular to AD; then as BD is perpendicular to the plane MN, the $\angle$ BDC will be a right angle; but this angle (Def. 3.) measures the inclination of the plane AB to MN; therefore these planes are perpendicular to each other.

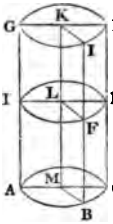
Cor. If the planes AB and CB are perpendicular to a third plane MN, their common section BD will also be perpendicular to MN.

For from the point D let a perpendicular be drawn to the plane MN; then this perpendicular must obviously lie in the plane A B, and for the same reason it must also lie in the plane CB, therefore it must form the common intersection of the planes.

85. THEOREM. If any prism ABCL be cut by a plane parallel to its base, the section will be equal to the base. (See figs. 1 and 2 to Art. **90.**)

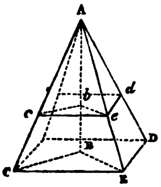
Let GHI be the parallel section. The two parallel planes ABC and GHI being cut by the plane ABGI, the line IG will be parallel to AB (Art. **80.**); in like manner GH will be parallel to BC, and so on to all the other sides; but, by the definition of a prism, AI and BG are parallel; therefore ABGI is a parallelogram, and, Art. **35.**, IG $=$ AB; in like manner GH $=$ BC, and so on to the other sides; that is, IGH and ABC are mutually equilateral. Again, by Art. **83.**, $\angle$ G $= \angle$ B, $\angle$ H $= \angle$ C, and so on to the other angles. Hence, it follows that the section ICH will be identical with ABC.

86. THEOREM. If a cylinder ABCG be cut by a plane parallel to its base, the section DFE will be a circle equal to the base.



Let ACHG and BIKM be two planes passing through the axis KM, cutting the section DFE in the points F, E, and L; then by the definition of a cylinder, BF is parallel to ML; and, Art. **80.**, LF is parallel to MB; therefore BFLM is a parallelogram, and then LF $=$ MB; and in the same manner it may be shown that LE $=$ MC, and so on. But MB is the radius of the circle ABC, consequently DFE is a circle equal to ABC.

87. THEOREM. In any pyramid CEDA, a section parallel to the base is similar to the base, and these two sections are to each other as the squares of their distances from the vertex.



Let *ced* be a section parallel to the base CED; draw A*b*B perpendicular to the two planes; join *eb* and EB; then, Art. **80.**, *ce* and *ed* will be parallel to CE and ED, and therefore, by Art. **83.** $\angle$ *ced* $= \angle$ CED; in like manner $\angle$ *d* $= \angle$ D, and so on; that is, the section *ced* is equiangular with CED. From the similar triangles ACE and A*ce* we have,

AE : A*e* :: CE : *ce*.

In like manner, from the similar triangles AED and A*ed*,

AE : A*e* :: ED : *ed*,

∴ CE : *ce* :: ED : *ed*.

In like manner it may be shown that all the sides in the section *ced* are proportional to all the corresponding sides in CED; hence we have (Art. 55.), area CED : area *ced* :: $CE^2 : ce^2$.

But, CE : *ce* :: AE : A*e*;

and also from the similar triangles ABE and A*be*,

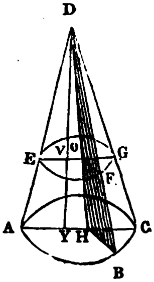
AE : A*e* :: AB : A*b*;

∴ CE : *ce* :: AB : A*b*, and squaring

$CE^2 : ce^2 :: AB^2 : Ab^2$,

∴ area CED : area *ced* :: $AB^2 : Ab^2$.

88. Theorem. In a cone ABCD, any section parallel to the base is a circle; and this section and the base are to each other, as the squares of their distance from the vertex.



Let DVY be a line perpendicular to the two parallel planes; and let the planes BHD and CHD pass through the axis DOH, cutting the section EFG in the lines OF and OG. Then by Art. **80.**, OF is parallel to HB, and OG to HC, and consequently the triangles DHB and DOF are similar, and also DHC and DOG; hence we have, DH : DO :: HB : OF; and DH : DO :: HC : OG;

∴ HB : OF :: HC : OG.

But HB=HC, being radii of the circle ABC, therefore OF=OG; and the same may be shown for any other point in the circumference EFG; therefore it is a circle.

Again, from the similar triangles DHY and DOV,

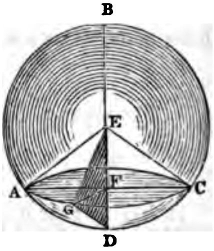
DY : DV :: DH : DO or :: HC : OG

∴ $DY^2 : DV^2 :: HC^2 : OG^2$; but, by *Cor.* Art. **75.**,

area circle ABC : area circle EFG :: $HC^2 : OG^2$,

∴ area circle ABC : area circle EFG :: $DY^2 : DV^2$.

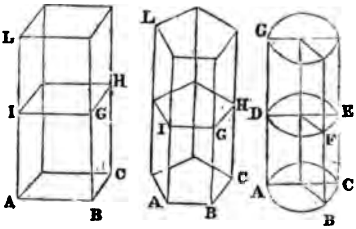
89. Theorem. Every section of a sphere by a plane is a circle.



Let AGC be the section of the sphere ADCB, of which E is the centre. From E draw EF perpendicular to the plane AGC, then BEFD will be the axis of the sphere. Let EADC and EGD be two planes passing through the axis. In the right-angled triangles AFE and GFE, we *have* EA = EG, being radii of the sphere, common to both triangles; therefore FA = FG. In like it may be shown that any other line drawn from F to umference of the section AGC is equal to FA; therefore a circle whose radius is FA.

The section of the sphere passing through the centre ve EA for its radius, which is obviously greater than nce AGC is called a *small* circle of the sphere, whereas on through the centre of the sphere is called a *great* All great circles of the sphere are evidently equal to ther.

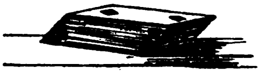
Theorem. Prisms and cylinders of equal bases and es, are equal to one another.



ose the prisms and r, in the annexed to be standing upon me plane, and let e cut by a plane to their bases, forming the sec- GH and DFE; then ections will be all o each, for, by Art. ey are respectively o their corresponding bases; and these latter are equal to her by hypothesis. In the same manner it may be shown y other parallel sections are equal. Now the prisms and r may be regarded as being made up of these indefinitely ual laminæ, or sections; and as there must be the same r in each solid, their altitudes being the same, it follows e prisms and cylinder are equal.

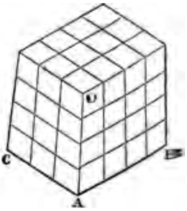
Illustrations and Applications.

pack of cards laid upon the table in the shape of a



will always form a solid of the same content or volume, whatever may be the inclination of the side faces. Here, the thin laminæ (the cards) form prisms, having the same base and altitude.

The method of proof given in this theorem is due to Cavallerius, and is known by the name of the *arithmetic of infinites.* It is certainly not so rigorous as the *method of exhaustions* given by Euclid; but the prolixity of this latter method renders it ineligible for elementary instruction, more especially as the method of the *integral calculus* is incomparably superior to all others in point of conciseness and precision.



2. *To find the content of a rectangular solid.* Let the length AB contain four units, the breadth AC three units, and the height AD four units; then in the whole height AD, we shall have 4 plates each one unit thick; but out of each plate we may evidently cut 12 solid units or cubes; therefore, in the whole rectangular solid we must have four times 12 or 48 solid units or cubes. Hence it appears that we find the number of solid units, or volume, in a rectangular solid, by multiplying the length, breadth, and thickness together, or what amounts to the same thing, by multiplying the area of the base by the perpendicular height. Thus,

$$\text{area base } ABC = 3 \times 4; \text{ and content } ABDC = 3 \times 4 \times 4.$$

If the linear dimensions are in inches, the content or volume will be in solid or cubic inches; if in feet, the volume will be in cubic feet, and so on.

Ex. 1. A block of stone is 6 feet long, 3 feet broad, and 2 feet thick; required its volume. *Ans.* 36 *cubic feet.*

Ex. 2. How many inch cubes can be cut out of a block 2 feet long, 7 inches broad, and 6 inches deep? *Ans.* 1008.

3. *To find the solid content of prisms and cylinders.* As *all prisms* and cylinders of equal bases and altitudes are

equal, it follows that the irregular prism ABCL as well as the cylinder ABCG, will have the same solid content as the rectangular solid ABCL; and hence the solid content of any of them will be equal to the area of the base multiplied by the perpendicular height.

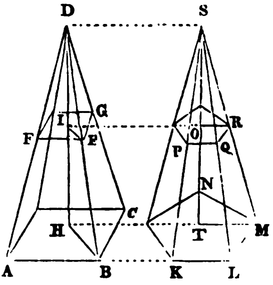
Ex. 1. If the area of the base of a prism is 24 square feet, and the perpendicular height 3 feet, required the solid content.

Here, area base $= 24$; $\therefore$ solidity $= 24 \times 3 = 72$ *c. ft.*

Ex. 2. The base of a prism is a right-angled triangle, whose perpendicular sides are 6 and 4 inches, and the altitude of the prism 5 feet; required the solidity.

Ans. 720 *cubic inches.*

91. THEOREM. Pyramids and cones of equal bases and altitudes, are equal to one another.



Suppose the pyramids, &c., to be standing upon the same plane, and let them be cut by a plane parallel to their bases, thereby forming the sections EFG and PQR; draw DIH perpendicular to the two planes EFG, ABC; and SOT perpendicular to the two planes PQR, KLM; then as DH = ST, the perpendicular DI = SO. But by Arts. 87. and 88.,

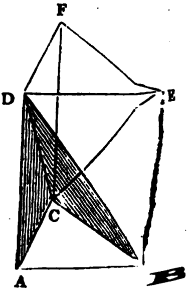
$$\text{area ABC} : \text{area EFG} :: \text{DH}^2 : \text{DI}^2,$$

$$\text{and area KLM} : \text{area PQR} :: \text{ST}^2 : \text{SO}^2,$$

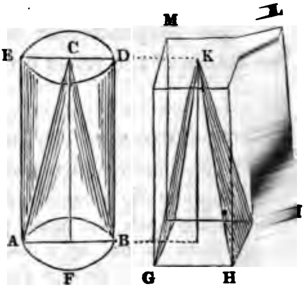
$$\therefore \text{area ABC} : \text{area EFG} :: \text{area KLM} : \text{area PQR};$$

but by hypothesis, area ABC $=$ area KLM; $\therefore$ area EFG $=$ area PQR. In the same manner it may be shown that any other parallel sections are equal; therefore the pyramids, being made up of these equal parallel sections, will be equal to each other. (See Art. **90.**)

92. THEOREM. Every pyramid having a triangular base is a third part of a prism, of the same base and altitude.



Let ABC and DEF be the two ends of the prism. Conceive sections to pass through BCD and DCE, then the prism will thus be cut into three equal pyramids. For, Art. **91.**, the pyramids ABDC and BEDC, standing on the equal bases ABD and BED, and having the same altitude, will be equal to each other. In like manner the pyramids ABDC and DFEC, standing upon the equal bases ABC and DFE, and having the same altitude, will also be equal to each other. Therefore the pyramid ABCD will be one-third the prism ABCF.



Cor. 1. Every pyramid GHIK is a third part of the prism GHIM of the same base and altitude; because if the base have four sides for example, it may be divided into triangles, and therefore the prism may be divided into two triangular prisms, and the pyramid into two pyramids, each of which is equal to one-third of its corresponding partial prism; therefore the two partial pyramids put together, or the entire pyramid, will be the third part of the two partial prisms put together, or the entire prism.

Cor. 2. Every cone AFBC is one-third part of a cylinder ABDE or of a prism GHIM, of equal base and altitude; because it has been proved that the cylinder ABDE is equal to the prism GHIM, and that the cone ABC is equal to the pyramid GHIK; but this pyramid is equal to one-third part of the prism GHIM; therefore the cone ABC will also be equal to one-third of the prism GHIM.

Applications.

To find the solid content of a cone or pyramid. Multiply the area of the base by the perpendicular height, and one-third the product will be the content.

Ex. 1. The base of a pyramid is a square, whose side is

3 feet, and the perpendicular height of the pyramid is 8 feet; required the solid content.

$$\text{Area base} = 3^2 = 9; \;\; \text{content} = \frac{9 \times 8}{3} = 24\,c.\,ft.$$

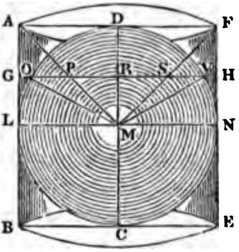
Ex. 2. The area of the base of a cone is 6 square feet, and the perpendicular height 7 feet; required the content.

Ans. 14 *cubic feet.*

Ex. 3. Required the content of a square pyramid, whose perpendicular height is 12 feet, and the side of the base 21 inches.

Ans. 12·25 *cubic feet.*

93. Theorem. A sphere is two-thirds of its circumscribing cylinder.



Let DC be the axis of the sphere CNDL, the circumscribing cylinder BEFA, and the cone AFM, where the point M is the centre of the sphere. Let GH be any section of the three solids, parallel to the base BE, cutting the cylinder in G, the sphere in O, and the cone in P; join these points and the centre M; and draw LN parallel to BE.

From the similar triangles ADM and PRM, we have,

$$\text{AD} : \text{DM} :: \text{PR} : \text{MR};$$

$$\text{but DM} = \text{AD}, \; \therefore \; \text{MR} = \text{PR}.$$

Again, from the right-angled triangle MRO, we have $\text{MO}^2 = \text{OR}^2 + \text{MR}^2$; but $\text{MO} = \text{GR}$, and $\text{MR} = \text{PR}$; $\therefore\ \text{GR}^2 = \text{OR}^2 + \text{PR}^2$. Now, Cor. Art. 75., the circle whose radius is r may be denoted by $r^2\pi$; therefore multiplying each side of the last equality by π, we have,

$$\pi\,.\,\text{GR}^2 = \pi\,.\,\text{OR}^2 + \pi\,.\,\text{PR}^2; \text{ that is,}$$

$$\text{area circle GH} = \text{area circle OV} + \text{area circle PS}.$$

In other words, the section of the cylinder is equal to the sum of the corresponding sections of the sphere and cone. Now, as this will be the case for every parallel section we can make, it follows that the volume of the semi-cylinder LF is equal to the sum of the volumes of the semi sphere LDN and the cone AFM; but the cone AFM is equal to one-third the semi-cylinder LF;

$\therefore$ Semi-cylinder LF $=$ semi-sphere LDN $+\frac{1}{3}$ semi-cylinder LF;

$\therefore$ $\frac{2}{3}$ semi-cylinder LF $=$ semi-sphere LDN; and taking the double of these equals,

$\frac{2}{3}$ the cylinder BEFA $=$ the sphere LDNO.

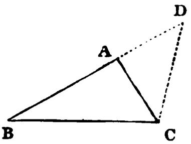
GEOMETRICAL EXERCISES.

94. The best mode of teaching the first principles of geometry is to proceed step by step from the truths that are known or admitted, until we at length arrive at the truths which were at first unknown, or which were required to be proved. This method, which has been almost invariably pursued in the preceding pages, is called *synthesis*. Although this plan is most admirably fitted for communicating a knowledge of truths which have been previously discovered, yet it is a comparatively inefficient instrument as it regards the *discovery* of truth, or the means by which a proposed truth may be established. There is another method of conducting geometrical investigations, called *analysis*, which consists in assuming the thing required to be proved, and then in tracing the various consequences which legitimately and progressively follow from this assumption, till we arrive at some conclusion which we already know to be true. When this is done, we at once infer that the assumption is also true, and our analysis is completed. If we now retrace the steps in the analysis, until we come to the thing assumed, we shall have pursued a synthetical demonstration. It will be observed that analysis is precisely the reverse of synthesis.

Geometrical analysis, being a method of invention, is peculiarly adapted for discovering the solution of problems. In conducting an analytical process, no general rules can be laid down: the nature of the problem will suggest various peculiar artifices to enable us to attain our object; sometimes lines will have to be drawn parallel to, or forming certain angles with, other lines; given angles or lines will have to be bisected; circles will have to be described, passing through certain points; and so on. The following exercises are intended to illustrate some of these artifices. The student may commence the exercises marked No. I. after he has read as far as p. 39. of this work; and those marked No. II. after he has read as far as p. 66.

No. I. — PROBLEMS.

1. Given the base, one of the angles at the base, and the sum of the remaining sides, to construct the triangle.



Analysis. — Suppose BAC the triangle required, having BC equal to the given base, $\angle$ B the given angle at the base, and BD equal to the sum of the sides BA and AC.

Join CD, then AC = AD, and ACD will be an isosceles triangle; therefore $\angle$ ACD = $\angle$ ADC.

Hence we have the following construction:

Take BC equal to the given base; draw BD making $\angle$ B equal to the given angle at the base; mark off BD equal to the sum of the two sides; join CD; and from C draw CA, meeting BD in A, making the $\angle$ ACD = $\angle$ ADC. Then BAC will be the triangle required.

2. Given the base, one of the angles at the base, and the perpendicular height, to construct the triangle.

3. To construct an isosceles triangle, whose perpendicular height is equal to the base.

4. Given the base and the angle at the vertex of an isosceles triangle, to construct it.

5. To find a *point* which shall have a given perpendicular

distance from a straight line, and shall also have a given distance from a given point in that line.

6. To find a point in a given straight line, which shall be at the same distance from two given points.

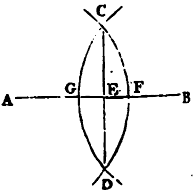
7. To construct a right-angled triangle, so that the hypotenuse shall be double the base. Show that the angle opposite to the base is equal to 30°.

8. To construct a square when the diagonal is given.

9. To construct a rectangle when the diagonal and one of the sides are given.

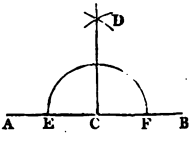
10. To divide a triangle into two equal parts by a line drawn from the vertex to the base.

11. To bisect a given straight line AB.



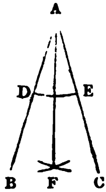
Synthesis.—With A as a centre, and any radius greater than half AB, describe the arc DFC; with B as a centre, and the same opening of the compasses, describe the arc DGC, cutting the former in the points C and D. Join CD, and it will bisect AB in the point E. Required the demonstration.

12. From a given point C in the straight line AB to draw the perpendicular CD.



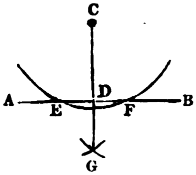
With C as a centre, and any radius CE, describe the semicircle EF. With E and F as centres, and any opening of the compasses greater than CE or CF, describe arcs cutting each other in D. Join CD, and it will be perpendicular to AB. Required the demonstration.

13. To bisect a given ∠ BAC.

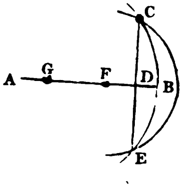


With A as a centre, and any opening of the compasses, describe the arc DE. With D and E as centres, and the same opening of the compasses, describe arcs cutting each other in F. Join AF, and it will bisect the ∠ BAC. Re*quired the demonstration.*

14. From a given point C to let fall the perpendicular CD upon the straight line AB.



With C as a centre describe an arc cutting AB in E and F. With E and F as centres, and any radius greater than half EF, describe two arcs cutting each other in G. Join CG, and it will be perpendicular to AB.



When the perpendicular falls near the end of the line. With any point G, in AB, as a centre, and GC as a radius, describe the arc CE. With any other point F, in AB, as a centre, and FC as a radius, describe another arc, cutting the former in the points C and E. Join CE, cutting AB in D, then CD will be perpendicular to AB. Required the demonstrations.

15. Through a given point to draw a straight line, which shall form an angle of 45° with a given line.

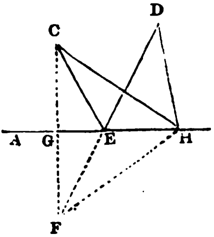
16. To find a point in the base of a triangle, from which a straight line drawn parallel to one of the sides and limited by the other, shall have a given length.

17. From a given point to draw a straight line, so that the part intercepted between two given parallel lines shall have a given length.

18. To divide a right angle into three equal parts.

19. To divide a parallelogram into two equal parts by a line passing through a given point.

20. To inscribe a square in a given right-angled isosceles triangle. Show that the square is half the triangle.

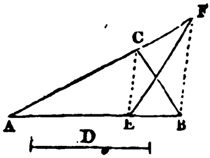


21. To find the point E in the straight line AH, so that the lines CE and DE drawn from the two given points C and D shall make the ∠ CEA = ∠ DEH.

Analysis. Suppose the thing to be done. Draw CG perpendicular to AE; and produce DE until it intersect CG produced in F. By Art. **28.** ∠ FEG = ∠ DEH, but ∠CEG = ∠DEH, ∴ ∠FEG = ∠CEG. Now in the triangles FEG and CEG, we have GE common to both, ∠FEG = ∠CEG, and ∠FGE = ∠CGE, being both right angles; therefore, by Art. 17., these triangles are identical, and GF = GC. Hence the following construction.

Draw CGF perpendicular to AH; take GF = GC; and join FD, cutting AH in E. Then E will be the point required.

22. The triangle ABC being given, to make the triangle AEF equal to it, having given the base AE equal to the line D.



Construction. Take AE = D; join CE; from B draw BF parallel to CE cutting AC produced in F; join EF, and AEF will be equal to ABC. Required the demonstration.

23. Given the base, one of the angles at the base, and the perpendicular height of a parallelogram, to construct it.

24. From a given isosceles triangle to cut off a trapezium, which shall have the same base as the triangle, and the three remaining sides equal to each other.

THEOREMS.

25. The diagonals of a parallelogram bisect each other.

26. The diagonals of a square divide it into four identical triangles.

27. The sum of the angles of a four-sided figure is equal to four right angles.

28. The two perpendiculars let fall from the extremities of the base of an isosceles triangle upon the sides, are equal to each other.

29. If a line be drawn bisecting a given angle, any point in that line will be equally distant from the two lines forming the angle.

30. Any straight line which bisects the diagonal of a parallelogram, also bisects the parallelogram.

31. If a given line be bisected by a perpendicular, any

point in this perpendicular will be equally distant from the extremities of the given line.

32. If a point be taken equally distant from two parallel lines, and any two straight lines be drawn through that point meeting the parallel lines, the two triangles thus formed will be identical.

33. If two straight lines are parallel to each other, a line drawn perpendicular to one of them, will also be perpendicular to the other.

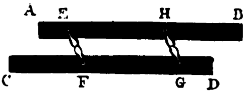
34. The perpendiculars drawn from the opposite angles of a parallelogram, upon the diagonal, are equal to each other.

35. If one of the equal sides of an isosceles triangle be produced beyond the vertex, the line which bisects the external angle will be parallel to the base.

36. If two straight lines bisect each other perpendicularly, the lines joining the extremities will form an equilateral parallelogram, or rhombus.

37. The area of a rhombus is equal to half the product of the two diagonals.

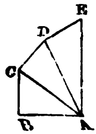
38. *The parallel ruler.* Two rulers AB and CD are connected by two equal pieces of brass, EF and HG, having pins or pivots at E, H, F, and G, so that EH = FG. Show that AB is always parallel to CD, whatever may be their distances from each other.



39. In Exercise 21, show that the sum of the lines CE and DE, is shorter than any other lines which can be drawn from the points C and D to meet the line AH.

No. II. — Problems.

40. To construct a square which shall be equal to the sum of the squares described upon the four given lines AB, BC, CD, and DE.



Construction. On AB and BC as sides construct the right-angled triangle ABC; on AC (the hypotenuse last found) and CD as sides construct the right-angled triangle ACD; and so on. Then the square described upon AE will be equal to the sum of the squares described upon AB, BC, CD, and DE. Required the demonstration.

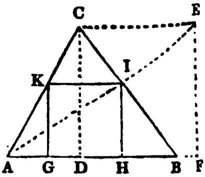
41. To inscribe an equilateral triangle in another equilateral triangle. Show that the inscribed triangle is one-fourth the other.

42. Through a given point, between two straight lines inclined to each other at a given angle, to draw a straight line which shall be bisected in that point.

43. Find a point in a given triangle, whose perpendicular distances from two of the sides are given.

44. To describe a circle, having a given radius, which shall pass through two given points.

45. To inscribe a square GKIH in a given triangle ACB.



Analysis. Suppose the thing to be done. Draw CD perpendicular to AB, and CE parallel to AB. Join AI and produce it until it cut CE in the point E. Then, because the triangles AKI and ACE are similar, we have $\frac{AC}{AK}=\frac{CE}{KI}$; but the triangles AKG and ACD are also similar, and $\frac{AC}{AK}=\frac{CD}{KG}$; $\therefore \frac{CE}{KI}=\frac{CD}{KG}$; but $KI=KG$, being sides of the square, $\therefore CE=CD$. Hence we have the following construction:

Draw CD perpendicular to AB, and CE parallel to AB. Take $CE=CD$, and join AE, cutting CB in I. Then I will be one of the corners of the square.

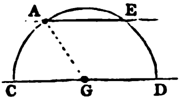
46. To draw a tangent to a circle, which shall be parallel to a given line.

47. In a given circle to draw a chord, which shall have a given length and shall be parallel to a given line.

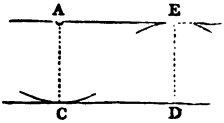
48. To describe a circle which shall pass through a given point, and also touch a given line in a given point.

49. To describe a circle which shall have a given radius, and its centre in a given straight line, and shall also touch another given straight line.

50. Through a given point A to draw a straight line parallel to a given straight line CD.



Take any point G, in CD, as a centre and the distance GA as a radius, and describe the semicircle CAD. Take the distance CA, and apply it from D to E, on the arc DE. Through the points A and E draw the straight line AE, and it will be parallel to CD.



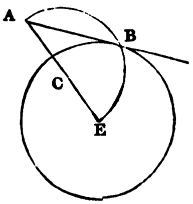
Or thus. From the given point A as a centre (with an opening of the compasses found by trial) sweep a circle touching the given line CD in C; with the same opening of the compasses and D as a centre, sweep the arc E; through A draw AE, just touching the arc E. Then AE will be parallel to CD.

Required the demonstrations.

51. To describe a circle which shall touch a given circle in a given point, and shall pass through a given point.

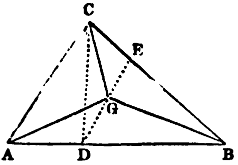
52. Given the base, the vertical angle, and the perpendicular height of a triangle, to construct it.

53. Through a given point A to draw a tangent AB to a given circle.



Construction. Join A and E, the centre of the given circle; bisect AE in C, and on C as a centre with CE or CA as a radius, describe the semicircle EBA, cutting the given circle in B; join AB, and this line will touch the given circle at B. Required the demonstration.

54. To divide a triangle ABC into three equal parts, by lines AG, BG, and CG, drawn from the three angles, to meet in a point G within the triangle.



Construction. Take AD a third part of AB; draw DE parallel to AC, the nearer side; bisect DE in G; join AG, BG, and CG, and these lines will divide the triangle into three equal parts. Required the demonstration.

55. To bisect a given triangle by a straight line drawn from a given point in one of the sides.

56. To bisect a given trapezium by a straight line drawn from one of the angles.

57. Inscribe a circle in a given quadrant.

THEOREMS.

58. Give a simple proof of the 47th of Euclid, when the sides of the right-angled triangle are equal to each other.

59. The square of the perpendicular of an equilateral triangle is equal to three-fourths of the square of the side.

60. Lines drawn to the middle of the sides of a square, will form a square which is half the given square.

61. If a straight line be drawn to the middle of the base of a triangle, it will bisect all lines drawn parallel to the base.

62. In a parallelogram the squares of the diagonals are equal to the sum of the squares of the sides.

63. The sum of the perpendiculars let fall from any point within an equilateral triangle upon the sides, is equal to the perpendicular let fall from one of its angles upon the opposite side.

64. If the sides of any four-sided figure be bisected, and these points be joined, the figure thus formed will be a parallelogram.

65. The two sides of a triangle are together greater than twice the line drawn from the vertex to the middle of the base.

66. The arcs intercepted between two parallel chords are equal.

67. If two circles cut each other, and any straight line be drawn through one of the points of intersection, limited by the circles, the lines drawn from the extremities of this line to the other point of intersection will contain a constant angle.

68. A line drawn from the middle of the side of a triangle, parallel to the base, will bisect the opposite side, and the line thus drawn will be equal to half the base.

69. The radius of the circle inscribed in an equilateral triangle is half the radius of the circumscribed circle.

PROBLEMS.

70. The base, the vertical angle, and the sum of the two sides, of a triangle being given, to construct it.

Analysis. Suppose BAC the triangle required (see fig. to Exercise 1.), having BC equal to the given base, $\angle$ BAC the given angle at the vertex, and BD equal to the sum of the sides BA and AC.

Join CD, then ACD will be an isosceles triangle; and therefore $\angle BAC = 2 \angle BDC$. Hence we have the following construction:

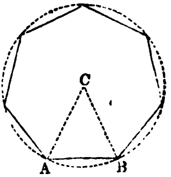
On the given base BC describe a circle so as to contain the $\angle$ BDC equal to half the given vertical angle. (Ex. 1. p. 56.) With B as a centre and a radius equal to BD, the sum of the two sides, describe an arc cutting the circle in D. Join CD and BD; draw CA making $\angle ACD = \angle ADC$. Then BAC will be the triangle required.

71. Given the sum of the sides, and the angles at the base of a triangle, to construct it.

72. To draw a straight line which shall touch two given circles.

73. To describe a circle, which shall touch a given straight line in a given point, and shall also touch a given circle.

74. To construct a regular polygon, on a given line AB, having any proposed number of sides.



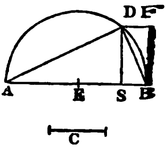
For example, let the number of sides be eight.

Analysis. Suppose the thing done, and that C is the centre of the circumscribed circle. Join AC and BC; then ACB will be an isosceles triangle, and $\angle A = \angle B$. Therefore $2\angle A + \angle C = 180^\circ$, and $\therefore \angle A = \frac{1}{2}(180^\circ - \angle C)$; but $\angle C = \frac{1}{8}$ of $360^\circ = 45^\circ$; $\therefore \angle A = \frac{1}{2}(180^\circ$

$-45°) = 67\frac{1}{2}°$. Hence the following construction. Draw A and BC, making the ∠ A and ∠ B each equal to $67\frac{1}{2}°$; on C as centre, with CA or CB as radius, describe a circle; apply the chor AB to the circumference, and it will form the polygon required.

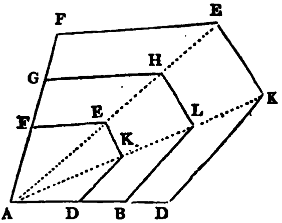
75. To inscribe a regular polygon of any proposed numbe of sides, in a given circle.

76. To divide a given straight line AB into two parts AS and SB, so that the rectangle contained by them shall be equal to the square upon the given line C.



Construction. Bisect AB in E, on E as a centre, with the radius EA or EB, describe the semicircle ADB. Draw BF perpendicular to AB, and make it equal to C. Draw FD parallel to AB, cutting the circle in D. From D let fall DS perpendicular to AB. Then $AS.SB = C^2$. Required the demonstration.

78. To make a rectilineal figure ADKEF similar to a given rectilineal figure ABLHG.



Join AL, AH, and produce them, if necessary. Take AD equal to the side of the required figure; draw DK parallel to BL; KE parallel to LH; and EF parallel to HG. Then ADKEF will be similar to ABLHG. For by Art. **31.**, ∠ AKD = ∠ ALB, and ∠ AKE = ∠ ALH; therefore by adding these equals, ∠ DKE = ∠ BLH; in precisely the same way it may be shown that ∠ KEF = ∠ LHG; and so on. Hence the figures are equiangular. Moreover from the similar triangles, we have, AK : AL :: DK : BL, and AK : AL :: KE : LH; therefore DK : BL :: KE : LH; in like manner it may be shown that KE : LH :: EF : HG, and so on. Hence the sides about the equal angles are proportional, and therefore the figures are similar.

Again, these similar figures are to each other as the squares of their like sides.

$$\text{For, } \frac{\text{area } AEF}{\text{area } AHG} = \frac{AE^2}{AH^2} = \frac{AK^2}{AL^2} = \frac{AD}{AB^2};$$

$$\therefore \frac{\text{area AEF}}{\text{AD}^2}=\frac{\text{area AHG}}{\text{AB}^2}.$$

In like manner,

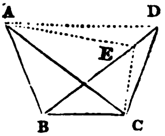
$$\frac{\text{area AKE}}{\text{AD}^2}=\frac{\text{area ALH}}{\text{AB}^2},\ \text{and}\ \frac{\text{area ADK}}{\text{AD}^2}=\frac{\text{area ABL}}{\text{AB}^2}.$$

Adding these equals together, we have,

$$\frac{\text{area ADKF}}{\text{AD}^2}=\frac{\text{area ABLG}}{\text{AB}^2};\ \therefore \frac{\text{area ADKF}}{\text{area ABLG}}=\frac{\text{AD}^2}{\text{AB}^2}.$$

THEOREMS.

79. The equal triangles, ABC and DBC, on the same base BC, are between the same parallels.



For if AD is not parallel to BC, draw AE parallel to BC, and join CE. Then, Art. **41.**, ΔABC $=\Delta$EBC; but by hypothesis, ΔABC $=\Delta$DBC; $\therefore$ ΔEBC $=\Delta$DBC, which is absurd; therefore AE is not parallel to BC. In the same manner it may be proved, that no other line but AD is parallel to BC.

80. In the triangle FCD, if $\text{FD}^2=\text{CD}^2+\text{CF}^2$; then $\angle$FCD is a right angle. (See fig. Art. **23.** p. 21.)

Draw CG perpendicular to CD; take CG $=$ CF; and join GD; then, Art. **45.**, $\text{GD}^2=\text{CD}^2+\text{CG}^2$ or CF^2; but $\text{FD}^2=\text{CD}^2+\text{CF}^2$; $\therefore$ $\text{GD}^2=\text{FD}^2$, or GD $=$ FD; hence, Art. **19.**, the triangles FCD and GCD are identical, and $\angle$FCD $=\angle$GCD $=$ a right angle.

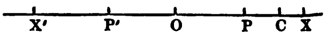
81. If the like sides of the triangles ABC and KFG be proportional, the triangles are equiangular. (See fig. Art. **54.** p. 46.)

Take Cr $=$ GK, and draw rm parallel to AB; then AC : CB :: Cr or GK : Cm; but by hypothesis, AC : CB :: GK : GF; $\therefore$ GF $=$ Cm; and in like manner KF $=rm$; that is, ΔKFG is identical with ΔrmG, and $\therefore$ equiangular with ΔABC.

APPLICATION OF ALGEBRA TO GEOMETRY.

95. In the preceding pages, various problems have been given, illustrating the application of algebra to geometry. In this section, we shall chiefly consider the geometrical interpretation of the plus and minus values of algebraic quantities.

Let it be required to construct the expression $a + b - c$, where each of these letters is put for a certain number of linear units. In the straight line XX′ take any fixed point O; measure off OC $= a$; CX $= b$; and XP $= c$; then OP $= a + b - c$. Here it will be observed, that whilst the plus quantities have been measured towards the right of the fixed point O, the minus quantity has been measured in the *contrary* direction. When c is greater than $a + b$, it is evident that the point required must lie to the left of O. In this case, let XP′ $= c$, then OP′ $= a + b - c =$ a minus quantity; thereby showing that if quantities measured to the right be called plus or +, those measured to the left must be minus or −. It is perfectly arbitrary as to the direction in which we measure plus quantities, provided we always take care to measure the minus quantities in the contrary direction. However, it is usual to refer plus quantities to the right, and minus ones to the left; and, in like manner, if we consider distances measured above a given line as plus, those that lie below that line must be minus.



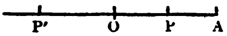
1. Let a traveller start from X towards O, at the rate of n miles per hour, what will be his distance from O, at the end of h hours, supposing XO $= m$ miles?

Take P for the required place, then the distance travelled, or XP, $= nh$, and therefore OP $= m - nh$.

If $m=20$, $n=4$, and $h=3$, then OP $=20-4\times 3=8$. result which shows that he had not arrived at the place and therefore OP must be regarded as a plus quantity.

Let now $m=20$, $n=4$, and $h=6$; then OP $=20-$ $\times 6=-4$. The question now arises, how are we to interret this minus result? It is obvious, from the data in this se, that the traveller must have passed the place O; that , he must have got 4 miles beyond it, and therefore P must taken at P′, lying 4 miles to the left off O.

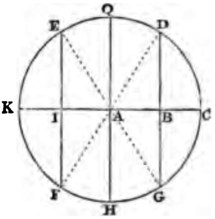
2. Let OA be a given line $=a$; it is required to find OP, so at OP^2 shall be equal to PA.



Let $x=$ OP, then PA $=a-x$; hence have the equation,

$$x^2=a-x, \text{ and } \therefore\ x=\tfrac{1}{2}\left(\pm\sqrt{4a+1}-1\right).$$

Let $a=6$, then $x=2$ or -3. Hence there are two points ich suit the conditions of the problem, viz. OP $=2$, and ′ $=3$; for we have the following verification: 1st, OP^2 PA, that is, $2^2=6-2$; 2d, OP′^{2} $=$ P′A, at is, $3^2=6+3$.



3. The radius AD of a circle is r; quired the position of a perpendicu-, DB $=p$, let fall upon the diameter .

Put $x=$ AB, then from the right-gled triangle ADB, we have,

$$AB^2=AD^2-DB^2, \text{ that is, } x^2=r^2-p^2,\ \therefore\ x=\pm\sqrt{r^2-p^2}.$$

$$\text{Let } r=5, \text{ and } p=3, \text{ then } x=\pm\sqrt{25-9}=\pm 4.$$

This result shows that we may draw the given perpendilar either to the right of A, or to the left of A. If we take $=$ AB $=4$, and draw the perpendiculars DB and EI, the iangle ABD will be the same as AIE, with this difference, that hereas the base of the former is measured towards the ght, the base of the latter is measured towards the left.

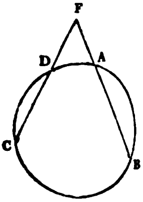
4. Given AD $=r$, and AB $=c$, in the last problem, to id BD.

Put $x=$ BD, then $x^2=r^2-c^2$, $\therefore\ x=\pm\sqrt{r^2-c^2}$.

Let $r=10$, and $c=6$; then $x=\pm\sqrt{100-36}=\pm 8$.

Here, if DB be produced to meet the circle in G, the line BD will be the plus value of x, and BG will be the minus one; thus the plus value is measured *upwards* from the line KC, and the latter *downwards* from the line KC.

5. Through a given point F, to draw a straight line FDC, cutting a given circle ABCD in the points D and C, so that the part DC may be equal to a given straight line.



Draw FAB cutting the given circle in A and B, and let FB $=a$, FA $=b$, DC $=c$, and FD $=x$. Then FC $=x+c$; and, by Art. **61.**, FC . FD $=$ FB . FA; that is, $(x+c)\,x=ab$.

Solving this equation, we find, $x=\frac{1}{2}\{-c\pm\sqrt{c^2+4ab}\}$.

Let $a=8$, $b=3$, and $c=2$; then $x=4$ or -6.

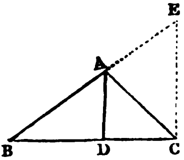
Here the positive value 4 is FD, in accordance with the assumption; and for the negative value we have FC $=6$. For if we put FC $=x$, we shall obtain an equation which has $x=6$ for its positive root.

Observation. The operations of algebra being more general than those of geometry, we frequently arrive at results, in the course of our analysis, which do not admit of direct geometrical interpretation, without changing the restricted form of the data. However, the meaning of a negative result may generally be ascertained by substituting $-x$ for x in the original equation, for then the resulting equation must have the same roots, only with their signs changed.

Thus, putting $-x$ for x, in the foregoing equation, we find $(-x+c)\times -x$, or $(x-c)\,x=ab$; $\therefore\ x=6$ or -4.

This equation is obviously the same as FD . FC $=$ FB . FA, x being put for FC.

6. Given the point A between the perpendicular lines CB and CE, to find a point B, so that BAE, passing through A, shall cut off the area BCE $= m^2$.



Draw AD perpendicular to CB, and let CD $= a$, AD $= c$, BC $= x$; then BD $= x - a$, and

$$\text{BD} : \text{AD} :: \text{BC} : \text{CE}$$

$$\therefore x - a : c :: x : \text{CE} = \frac{cx}{x-a};$$

$$\text{but BC . CE} = 2m^2, \therefore x \times \frac{cx}{x-a} = 2m^2;$$

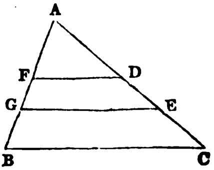
$$\therefore x = \frac{m^2}{c} \pm \frac{m}{c}\sqrt{m^2 - 2ac}.$$

Here the two values of x are always plus and real, when m^2 is greater than $2ac$. When $2ac$ is greater than m^2, the problem is impossible, for we have then to find the square root of a negative quantity. When A is taken on the line CE, then CD or $a = 0$, and $x = \frac{2m^2}{c}$.

Let $a = 5$, $c = 2$, and $m^2 = 36$, then

$$x = \frac{36}{2} \pm \frac{6}{2}\sqrt{36 - 2 \times 5 \times 2} = 30 \text{ or } 6.$$

7. In the given triangle GAE, to find the position of a given straight line FD, which shall be parallel to the base GE.



Let GE $= a$, GA $= b$, FD $= m$, and GF $= x$; then FA $= b - x$, and by the similar triangles GAE and FAD,

$$\text{GE} : \text{GA} :: \text{FD} : \text{FA},$$

$$\text{or } a : b :: m : b - x;$$

$$\therefore mb = a(b - x); \therefore x = \frac{b(a-m)}{a}.$$

Let $b = 8$, $a = 4$, and $m = 2$, then $x = 4$. This result, being positive, shows that GF is measured towards the upper side of the line GE, agreeably to the assumption.

Let $b = 12$, $a = 3$, and $m = 5$, then $x = -8$. This result, being minus, indicates that x must be measured in a direction contrary to that in which it was measured in the first case. Take $GB = 8$, and draw BC parallel to GE, then BC will be the position of the given line in this case. For, changing x into $-x$ in the preceding proportion, we have, $a : b :: m : b + x$, a proportion which applies to the triangles AGE and ABC.

8. The base and perpendicular of the triangle, in Exercise 45, p. 88., being given, to find the side of the inscribed square.

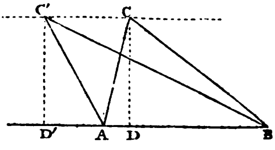
Let $AB = b$, $CD = p$, and the side of the square $GH = x$. Then by the similar triangles ACB and KCI, we have,

$$AB : CD :: KI : CD - KG,$$

$$\text{or, } b : p :: x : p - x,$$

$$\therefore px = b\,(p - x), \quad \therefore x = \frac{bp}{b + p}.$$

9. Given the base of a triangle, its perpendicular height, and the ratio of its other sides, to find the triangle.



Suppose ABC to be the required triangle, and CD the given perpendicular. Let $AB = 5$, $CD = 2$, $BC = 2\,AC$, and $AD = x$. Then $DB = 5 - x$, and from the right-angled triangles ADC and BDC, we have,

$$AC^2 = x^2 + 4\,; \text{ and } BC^2 = (5 - x^2) + 4\,;$$

but $BC = 2AC$, or $BC^2 = 4AC^2$; $\therefore (5 - x)^2 + 4 = 4\,(x^2 + 4)$.

Hence we find $x = 1$ or $-4\frac{1}{3}$. Here there are two distinct triangles answering to the conditions of the problem, *viz.*, ABC and ABC′, where $AD = 1$, and $AD' = 4\frac{1}{3}$.

10. The base of a right-angled triangle $= 6$, and the

hypotenuse added to twice the perpendicular $= 26$; required the perpendicular. *Ans.* 8.

11. The hypotenuse of an isosceles right-angled triangle is 18; required the sides. *Ans.* 12·72.

12. The radius of a quadrant of a circle is r; required the radius of the inscribed square. *Ans.* $\frac{r}{\sqrt{2}}$.

13. Required the side of the square inscribed in a segment of a circle, when the radius of the circle is 5, and the chord of the segment 8. *Ans.* 1·908, or $-6·708$.

14. The radius of a quadrant of a circle is r; required the radius of the inscribed circle. *Ans.* $\frac{r}{1+\sqrt{2}}$.

15. Given the perimeter, or sum of the sides, of a right-angled isosceles triangle $= p$; to find the hypotenuse. *Ans.* $p\,(\sqrt{2}-1)$.

16. Given the difference between the perimeter and perpendicular of an equilateral triangle $= d$; to find the sides. *Ans.* $\frac{2d}{6-\sqrt{3}}$.

17. The base of a right-angled triangle is 12, and the difference of twice the hypotenuse and the perpendicular is 21; required the perpendicular. *Ans.* 5 *or* 9.

18. The sum of the two sides of a triangle is s, and the two segments of the base, formed by the line bisecting the vertical angle, are c and d respectively; required the sides. *Ans.* $\frac{cs}{c+d}$ *and* $\frac{ds}{c+d}$.

19. Given the area of a triangle $= 160$, the base $= 40$, and one of the sides $= 20$; to find the remaining side. *Ans.* 23·099, or 58·876.

20. The two legs of a right-angled triangle are 12 and 16; required the sides of the inscribed rectangle whose area is equal to half the area of the triangle. *Ans.* 6 *and* 8.

21. The three perpendiculars drawn from a point within

an equilateral triangle, upon the sides, are 4, 6, and 8; and the area of the triangle is 81; required the sides. *Ans.* 9

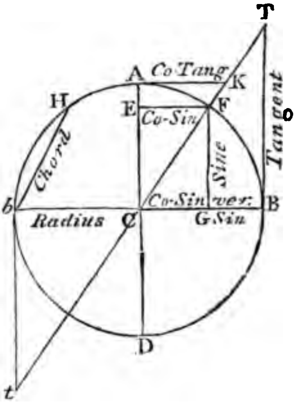
22. I took a point, in the continuation of the diagonal of a square, and found its distance from the nearest corners of the square to be 40 and 60; required the side of the square. *Ans.* 24·63.

TRIGONOMETRY.

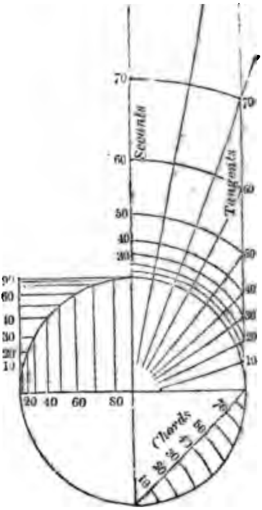
DEFINITIONS AND CERTAIN FUNDAMENTAL THEOREMS.

1. TRIGONOMETRY is that branch of mathematics which treats of the relations subsisting between the sides and angles of triangles.

2. Let BAD be a circle described upon the centre C, with the radius CB = 1 or unity. Draw the diameter BC*b*, and let BF be any arc of the circle, then the number of degrees in this arc will be the measure of the angle BCF (Art. 64.). Thus if the arc BF contain 60° the ∠BCF will be an angle of 60°. From F let fall FG perpendicular to CB, then FG is called the *sine* of the arc BF or ∠BCF, CG is called the *cosine*, and BG the versed sine. Hence the following definition will be readily understood; *the sine of an arc is the perpendicular let fall from one extremity of the arc upon the radius passing through the other extremity; and the cosine is the distance from the centre to the foot of the sine.* From B draw BT perpendicular to CB, and produce CF until it intersect BT in the point T, then BT is called the *tangent* of the arc BF or ∠BCF, and CT is the *secant.* Through the centre C draw AD perpendicular to CB, then BA will be an arc of 90°, and the arc AF will be the *complement* of the arc BF. From F let fall FE perpendicular to CA, then EF or CG will be the cosine of the arc BF; draw AK perpendicular to AC, cutting



CT in the point K, then AK is called the *cotangent* of the arc BF or ∠BCF, and CK is the *cosecant.*



The annexed figure shows the method of constructing a scale of chords, sines, tangents, and secants. The most useful part of this scale is the line of chords. In order to show the use of this scale, let it be required to draw BG, from a given point B in the straight line BF, making the ∠B = 50°. (See *fig.* p. 22.). With B as a centre and the chord of 60° as a radius (see **Art. 68.** Geo.) describe the arc FG; with the compasses take off the chord of 50°, and on F as a centre describe an arc cutting the former in the point G; join BG, and then ∠FBG = 50°, as required.

Tables have been calculated giving the sines, cosines, &c., of all angles, so that the sine, cosine, &c., of an angle having any number of degrees and minutes, can at once be found, and conversely also the number of degrees and minutes corresponding to any given quantity representing the sine, cosine, &c. The method of calculating these tables will be hereafter explained.

These definitions may be expressed in the following abbreviated form. Let the number of degrees in the arc BF or ∠BCF = A, then FG = sin A, CG = cos A, BT = tan A, CT = sec A, AK = cot A, CK = cosec A, and BG = v. sin A.

From these definitions we readily derive the following properties.

3. Since FG = CE, it appears that while FG is the sine of the arc BF, it is at the same time the cosine of the arc AF, *that is, the sine of an arc = the cosine of its complement;* or putting A for the number of degrees in the arc BF, $\sin A = \cos(90° - A)$; thus, for example, $\sin 30° = \cos 60°$. In like manner, AK is the tangent of the arc AF, and at the same time the cotangent of the arc BF; hence also *the tangent of an arc = the cotangent of its complement.* And so on to the other cases.

4. If the arc KE be taken equal to CD (see *fig.* page 95.), then $EI = DB$, and arc CE is the supplement of arc CD; hence *the sine of an arc = sine of its supplement*, or putting A for the arc CD, $\sin A = \sin(180° - A)$. Thus $\sin 40° = \sin 150°$. So that in finding the sine of an arc greater than 90° from the tables, we must seek for the sine of its supplement.

5. From the right-angled triangle CFG (see *fig.* p. 101.), we have $FG^2 + CG^2 = CF^2$; now when $CF = 1$, $FG =$ sine arc BF or $\sin A$, and $CG =$ cosine arc BF or $\cos A$,

$$\therefore \ \sin^2 A + \cos^2 A = 1 \ \ldots\ldots \ (1).$$

From the similar triangles CFG and CTB, we have

$$\frac{BT}{CB} = \frac{GF}{CG}, \text{ that is, } \tan A = \frac{\sin A}{\cos A} \ \ldots \ (2)$$

$$\frac{CT}{CB} = \frac{CF}{CG}, \text{ that is, } \sec A = \frac{1}{\cos A} \ \ldots \ (3)$$

$$\frac{FG}{CF} = \frac{BT}{CT}, \text{ that is, } \sin A = \frac{\tan A}{\sec A} \ \ldots \ (4).$$

From the similar triangles CBT and CAK, we have,

$$\frac{BT}{CB} = \frac{CA}{AK}, \text{ that is, } \tan A = \frac{1}{\cot A} \ \ldots \ (5)$$

$$\frac{CK}{CA} = \frac{CT}{BT}, \text{ that is, } \operatorname{cosec} A = \frac{\sec A}{\tan A} \ \ldots \ (6).$$

From the right-angled triangle CBT, we have,

$$CT^2 = CB^2 + BT^2, \text{ that is, } \sec^2 A = 1 + \tan^2 A \ \ldots \ (7).$$

6. From these equations any of the quantities $\sin A$, $\cos A$, $\tan A$, &c., may be found when any one of them is given.

For example, from (1) we have

$$\sin A = \sqrt{1 - \cos^2 A}, \text{ and } \cos A = \sqrt{1 - \sin^2 A} \quad \ldots \ (1).$$

Having given the tangent let the cosine be required.

From eq. (3) we have, $\cos A = \frac{1}{\sec A}$; substituting the value of sec A derived from eq. (7), we find,

$$\cos A = \frac{1}{\sqrt{1 + \tan^2 A}} \quad \ldots \ (2).$$

To find the sine in terms of the tangent.

In eq. (4) substitute the value of sec A derived from eq. (7). and we at once obtain,

$$\sin A = \frac{\tan A}{\sqrt{1 + \tan^2 A}} \quad \ldots \ (3).$$

7. From the figure, p. 95., it will readily be seen that the chord DBG is twice DB, and also the arc DCG is twice the arc DC; but DB is the sine of the arc DC,

∴ *the chord of an arc = twice the sine of half the arc.*

8. To find the numerical values of the sine, cosine, and tangent, of 30°, 60°, and 45°.

1. By the preceding theorem, the chord of $60° = 2 \sin 30°$; but by Art. **68.** Geo., the chord of $60° = \text{radius} = 1$;

$$\therefore\ 2 \sin 30° = 1, \text{ and } \sin 30° = \tfrac{1}{2}.$$

$$\text{By Art. } \mathbf{6.},\ \cos 30° = \sqrt{1 - \sin^2 30} = \frac{\sqrt{3}}{2}.$$

$$\text{By eq. (2), Art. } \mathbf{5.}\ \tan 30° = \frac{\sin 30°}{\cos 30°} = \frac{1}{2} \div \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{3}}.$$

2. By Art. **3.** $\sin 60° = \cos 30° = \frac{\sqrt{3}}{2}$,

$$\text{and } \cos 60° = \sin 30° = \tfrac{1}{2}.$$

$$\tan 60° = \frac{\sin 60°}{\cos 60°} = \frac{\sqrt{3}}{2} + \frac{1}{2} = \sqrt{3}.$$

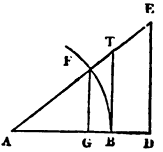
3. By Art. 3. $\sin 45° = \cos 45°$; but by Art. 5., $\sin^2 45° + \cos^2 45° = 1$;

$$\therefore\ 2\sin^2 45° = 1,$$

$$\therefore\ \sin 45° = \frac{1}{\sqrt{2}} = \cos 45°.$$

$$\tan 45° = \frac{\sin 45°}{\cos 45°} = 1.$$

9. *Extension of the definitions of sines, cosines, &c.*



Let BF be an arc whose radius AB is unity, FG the sine, AG the cosine, and BT the tangent. From any point D, in the line AB produced, draw DE perpendicular to AD, intersecting AF produced in the point E; then from the similar triangles AGF, ABT, and ADE, we have,

$$\frac{\text{GF}}{\text{AF}} = \frac{\text{DE}}{\text{AE}}, \text{ that is, } \sin \text{A} = \frac{\text{DE}}{\text{AE}}.$$

This result shows that, *the sine of the angle at the base of a right-angled triangle is equal to the perpendicular divided by the hypotenuse.*

$$\text{Again } \frac{\text{AG}}{\text{AF}} = \frac{\text{AD}}{\text{AE}}, \text{ that is, } \cos \text{A} = \frac{\text{AD}}{\text{AE}}.$$

This result shows that, *the cosine is equal to the base divided by the hypotenuse.*

$$\frac{\text{BT}}{\text{AB}} = \frac{\text{DE}}{\text{AD}}, \text{ that is, } \tan \text{A} = \frac{\text{DE}}{\text{AD}}.$$

This result shows that, *the tangent is equal to the perpendicular divided by the base.*

And so on to other trigonometrical definitions. We shall therefore hereafter use these results, when occasion requires, as the expressions of the sines, cosines, &c.

10. Let r be put for AE, then DE and AD will be respectively the sine and cosine of an arc whose radius is r. Hence the expressions, $\sin A = \frac{DE}{AE}$, and $\cos A = \frac{AD}{AE}$ become,

$$\sin A \text{ to rad. unity} = \frac{\sin A \text{ to rad. } r}{r}$$

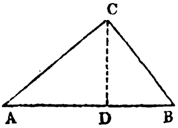
$$\cos A \text{ to rad. unity} = \frac{\cos A \text{ to rad. } r}{r}.$$

And so on to other cases. To transform formulæ, therefore, having the radius unity, into others having the radius r, we have simply to substitute $\frac{\sin A}{r}$ for $\sin A$, $\frac{\cos A}{r}$ for $\cos A$, &c.

Moreover if we have given the values of sin A, cos A, to radius unity, we can find them for radius r by simply multiplying by r.

11. *The ratio of any two sides of a triangle is equal to the ratio of the sines of the opposite angles.*

For by Art. **9.**

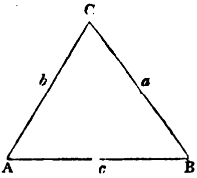


$$\sin A = \frac{CD}{AC}, \text{ and } \sin B = \frac{CD}{BC};$$

dividing the first equation by the second,

$$\frac{\sin A}{\sin B} = \frac{CD}{AC} \div \frac{CD}{BC}, \quad \therefore \quad \frac{BC}{AC} = \frac{\sin A}{\sin B}.$$

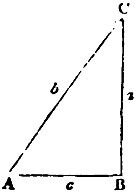
Putting a for the side opposite the angle A, b for the side opposite the angle B, and c for the side opposite to the angle C, then



$$\frac{a}{b} = \frac{\sin A}{\sin B} \quad \ldots (1)$$

And so on to any other sides. Any three terms in this equation being given, the remaining one *may be found.*

In the right-angled triangle ABC we simply have by Art. 9.



$$\sin A = \frac{a}{b} \ldots (2)$$

$$\cos A = \frac{c}{b} \ldots (3)$$

$$\text{and } \tan A = \frac{a}{c} \ldots (4).$$

Any two terms in these equations being given, the remaining one may be found.

These properties will enable us to solve a variety of useful problems.

EXAMPLES.

Solution of Triangles.

Ex. 1. In the triangle ABC, given the side AC or $b = 200$, the $\angle B = 58^\circ 35'$, and $\angle A = 64^\circ$; to find the side BC or a.

Here by eq. (1), we have, $\frac{a}{200} = \frac{\sin 64^\circ}{\sin 58^\circ 35'}$,

$$\therefore \text{BC or } a = \frac{\sin 64^\circ \times 200}{\sin 58^\circ 35'} = \frac{\cdot 89879 \times 200}{\cdot 8534} = 210 \cdot 6.$$

2. To find AB or c, in the last example.

In this case, we have, $\frac{c}{b} = \frac{\sin C}{\sin B}$. Now, by Art. 34., $\angle C = 180^\circ - (\angle B + \angle A) = 57^\circ 25'$.

$$\therefore c = \frac{\sin 57^\circ 25' \times 200}{\sin 58^\circ 35'} = 197 \cdot 4.$$

3. Given as in the last example, $c = 80$, $\angle A = 80^\circ 5'$, $\angle B = 43^\circ$, and $\therefore \angle C = 56^\circ 55'$; to find a and b.

Ans. BC $= 94 \cdot 05$ *and* AC $= 65 \cdot 11$.

4. Given $c = 70$, $b = 80$, and $\angle B = 60^\circ$; to find the remaining angles and side.

Here we must first find the $\angle C$; for this purpose, we ha

$$\frac{c}{b}=\frac{\sin C}{\sin B},$$

$$\therefore \sin C=\frac{\sin 60^\circ \times 70}{80}=\cdot 75777,$$

$$\therefore \text{ from the tables, we find, } \angle C=49^\circ\ 16',$$

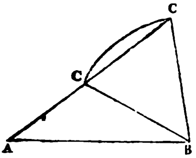
$$\text{and } \therefore\ \angle A=180^\circ-(\angle B+\angle C)=70^\circ\ 44'.$$

Then, proceeding as in the foregoing examples, we fi $a=87{\cdot}2$.

5. Given $c=162$, $b=270$, and $\angle B=90^\circ$; to find th remaining angles and side.

Ans. $\angle C=36^\circ\ 52'$, $\angle A=53^\circ\ 8'$, *and* $a=216$.

6. In the triangle ABC, given $AB=345$, $BC=232$, and $\angle A=37^\circ\ 20'$; to find $\angle C$ and the side AC.



This question admits of two answers. The reason of this will be best understood by going over the construction. From a scale of equal parts take off $AB=345$; draw AC, making the $\angle A=37^\circ\ 20'$; with the centre B, and radius $BC=232$, describe an arc cutting AC in the two points C and C; join BC. Then AC being measured off the scale of equal parts, will give the two values, 174 and 374, for the side required.

By calculation. First to find $\angle C$, we have $\frac{c}{a}=\frac{\sin C}{\sin A}$;

$$\therefore \sin C=\frac{\sin 37^\circ\ 20' \times 345}{232}.$$

Hence we find $\angle C=64^\circ\ 24'$ or $115^\circ\ 36'$, because (Art. 4.) the sine of any angle is the same as the sine of its supplement. We now find (Art. 34.) $\angle B=78^\circ\ 16'$ or $27^\circ\ 4'$.

Taking the first angle and proceeding as in *Ex.* 3. we find, $AC=374{\cdot}5$. In like manner, taking the second angle, *we find,* $AC=174{\cdot}07$.

7. Given as in the last example, AB $= 90$, BC $= 75$, and $\angle$ A $= 45°$; to find $\angle$ C and the side AC.

Ans. 58° 3′ *or* 121° 57′, *and* AC = 103·3 *or* 23·5.

8. In the right-angled triangle ABC, given the hypotenuse $b = 48$, and $\angle$ A $= 53° \; 8'$; to find the perpendicular and base.

First, to find the perpendicular, we have, by eq. (2), $\sin \text{A} = \frac{a}{b}$, that is in the present case, $\sin 53° \; 8' = \frac{a}{48}$,

$$\therefore \; a = \sin 53° \; 8' \times 48 = 38{\cdot}4.$$

Second, to find the base, we have, by eq. (3), $\cos \text{A} = \frac{c}{b}$, that is, $\cos 53° \; 8' = \frac{c}{48}$,

$$\therefore \; c = 28{\cdot}8.$$

9. Given as in the last example, $b = 64{\cdot}5$, and $\angle$ A $= 39° \; 10'$; to find a and c. *Ans.* 50 *and* 40·73.

10. In the right-angled triangle ABC, given $c = 200$, and $\angle$ A $= 35°$, to find a.

Here by eq. (4), $\tan \text{A} = \frac{a}{c}$, that is, $\tan 35° = \frac{a}{200}$,

$$\therefore \; a = \tan 35° \times 200 = 140.$$

11. Given as in the last example, $c = 100$ ft., and $\angle$ A $= 74°$; to find a. *Ans.* 348·7 *ft.*

12. *To find the area of the triangle* ABC, *having given the two sides* AB *and* AC, *and the included angle* A. (See fig. Art. 11., p. 106.)

$$\text{Area triangle ABC} = \tfrac{1}{2} \times \text{AB} \times \text{CD}.$$

In this equation it is requisite, that we should eliminate CD; for this purpose, we have, Art. 9.,

$$\sin \text{A} = \frac{\text{CD}}{\text{AC}},$$

$$\therefore \; \text{CD} = \text{AC} \times \sin \text{A},$$

$$\therefore \; \text{area triangle ABC} = \tfrac{1}{2} \times \text{AB} \times \text{AC} \times \sin \text{A};$$

that is, *the area of a triangle is equal to half the product of any two sides by the sine of the included angle.*

Since a parallelogram is double the area of the triangle cut off by a diagonal, it follows that, *the area of a parallelogram is equal to the product of the two adjacent sides by the sine of the included angle.*

EXAMPLES.

Ex. 1. The two sides of a triangle are respectively 1 and 30 ft., and the included angle 30°.

1. In the triangle ABC (see fig. Art. 11.), AB = 30 ft., AC = 10 ft., and ∠ A = 30°; required the area.

$$\text{Here the perpendicular CD} = 10 \times \sin 30° = 5,$$

$$\therefore \text{ area triangle} = \tfrac{1}{2} \times 30 \times 5 = 75 \text{ sq. ft.}$$

2. Required the same as in the last problem, when AB = 40 yds., AC = 20 yds., and ∠ A = 45°. *Ans.* 282·8 *sq. yds.*

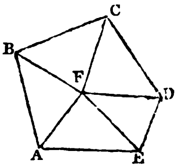
3. Required the area of a parallelogram, whose adjacent sides are 36 and 25·5 feet, and the included angle 58°.

$$\text{Here, area} = 36 \times 25{\cdot}5 \times \sin 58° = 778{\cdot}5 \text{ sq. ft.}$$

4. Required the area of a parallelogram, whose adjacent sides are 5·07 and 10·4, and the included angle 30°.

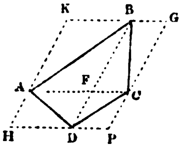
Ans. 26·36.

5. To measure a large piece of ground ABCDE, the surveyor places his instrument at a point F, from which he can see all the corners. He finds ∠ AFB = 85°, ∠ BFC = 80°, ∠ CFD = 86°, ∠ DFE = 70°, and the remaining ∠ EFA = 39°, which it must be in order to make up 360°. He then measures FA = 2000 links, FB = 1640 links, FC = 2310 links, FD = 3000 links, and FE = 1500 links. Required the area.



Ans. 100 *acres,* 22·4 *poles.*

13. *To find the area of a trapezium* ABCD, *having given the* ∠ AFD *formed by the diagonals* AC *and* BD, *and the lengths of these diagonals.*



Draw HAK and PCG parallel to BD, and HDP and KBG parallel to AC; then it is obvious from the figure, that the parallelogram HKGP, thus formed, will be double the area of the trapezium ABCD, and also that ∠ H = ∠ AFD or ∠ F; but by Art. 12.,

$$\text{Area HKGP} = \text{HK} \times \text{HP} \times \sin \text{H} = \text{BD} \times \text{AC} \times \sin \text{F};$$

$$\therefore \text{ area ABCD} = \tfrac{1}{2} \times \text{BD} \times \text{AC} \times \sin \text{F}.$$

Ex. 1. In measuring the four-sided field ABCD, the surveyor placed his theodolite at the point F, where the diagonals of the field intersected each other; he then found the ∠ F = 34° 15′, the diagonal BD = 600 links, and AC = 420 links; required the area of the field.

Here from the table of nat. sines, we find, sin 34° 15′ = ·562805,

$\therefore$ area ABCD $= \frac{1}{2} \times 600 \times 420 \times \cdot 562805 = 70913{\cdot}4$ sq. links = 2 *r.* 33·46 *p.*

2. Required the same as in the last problem, when ∠ F = 26° 25′, BD = 800 links, and AC = 720 links.

Ans. 1 *acre,* 1 *r.* 5 *p.*

Computation by Logarithms.

14. Calculations in trigonometry are very much facilitated by the aid of logarithms.

In the *common* logarithms, the logarithm of a number is that index or power of 10 necessary to produce the number; thus as $100 = 10^2$, the logarithm of 100 is 2; $1000 = 10^3$, the logarithm of 1000 is 3; or if N be any number such that $N = 10^x$, then x is the logarithm of N, or $\log N = x$. Here the 10 is called the base of the system. In the hyperbolic system the base is 2·71828, which number is usually represented by e.

15. Since $1 = 10^0$, $\therefore \log 1 = 0$; $10 = 10^1$, $\therefore \log 10 = 1$; $100 = 10^2$, $\therefore \log 100 = 2$; $1000 = 10^3$, $\therefore \log 1000 = 3$; and so on. Hence it follows that the logarithm of a number between 1 and 10 will be some decimal quantity; the log. of any number between 10 and 100 will be between 1 and 2, or 1 + a fraction; the log. of any number between 100 and 1000 will be between 2 and 3, or 2 + a fraction; and so on generally, the log. of any number having n integral figures will contain $n - 1$ units + a fraction. The whole number in the logarithm is called the *index* or *characteristic*, and the decimal part is called the *mantissa;* thus it has been found that $406{\cdot}5 = 10^{2{\cdot}60906}$, or $\log 406{\cdot}5 = 2{\cdot}60906$, where 2 is the characteristic and ·60906 is the mantissa. As the characteristic is always 1 less than the number of figures in the integral part of the number, the logarithm given in the tables is only the decimal part of the required logarithm, and the integral part or characteristic must therefore be supplied by the calculator.

16. The logarithm of the product of any numbers is found by adding together the logarithms of the factors.

Let A, B, C, &c. be any numbers, and x, y, z, &c. their logarithms, taking a as the base; then,

$$\text{A} = a^x,\ \text{B} = a^y,\ \text{C} = a^z,\ \&\text{c.};\ \text{whence by multiplication,}$$

$$\text{A} \times \text{B} \times \text{C} \times \&\text{c.} = a^x \times a^y \times a^z \times \&\text{c.} = a^{x+y+z+\,\&\text{c.}}$$

$\therefore \log (\text{A} \times \text{B} \times \text{C} \times \&\text{c.}) = x + y + z + \&\text{c.} = \log \text{A} + \log \text{B} + \log \text{C} + \&\text{c.}$

Ex. Find by logarithms the value of $36 \times 2{\cdot}45$.

$$\begin{aligned} \text{Here } \log (36 \times 2{\cdot}45) &= \log 36 + \log 2{\cdot}45 \\ \log 36 &= 1{\cdot}5563 \\ \log 2{\cdot}45 &= \underline{\ \ {\cdot}38917} \\ & 1{\cdot}94547 = \log 88{\cdot}20 \end{aligned}$$

$$\therefore 36 \times 2{\cdot}45 = 88{\cdot}2.$$

17. The logarithm of the quotient of two numbers is found by subtracting the logarithm of the divisor from the logarithm of the dividend.

$$\text{For } \frac{A}{B} = \frac{a^x}{a^y} = a^{x-y}$$

$$\therefore \log\frac{A}{B} = x - y = \log A - \log B.$$

Ex. Divide 782 by 23.

$$\text{Here } \log\frac{782}{23} = \log 782 - \log 23$$

$$\begin{array}{rl} \log 782 = & 2{\cdot}8932 \\ \log \ \ 23 = & 1{\cdot}36172 \\ \hline & 1{\cdot}53148 = \log 34 \end{array}$$

$$\therefore 782 \div 23 = 34.$$

18. The logarithm of a number raised to any power is found by multiplying the logarithm of the number by the exponent.

$$\text{For } A = a^x, \ \therefore A^n = a^{nx}$$

$$\therefore \log A^n = nx = n \times \log A,$$

$$\text{Or in general } A^{\frac{n}{m}} = a^{\frac{nx}{m}};$$

$$\therefore \log A^{\frac{n}{m}} = \frac{nx}{m} = \frac{n}{m}\log A.$$

Ex. 1. Calculate $1{\cdot}2^3$ by logarithms.

$$\log 1{\cdot}2^3 = 3 \times \log 1{\cdot}2$$

$$\begin{array}{rl} \log 1{\cdot}2 = & {\cdot}07918 \\ & 3 \\ \hline & {\cdot}23754 = \log 1{\cdot}728 \end{array}$$

$$\therefore 1{\cdot}2^3 = 1{\cdot}728.$$

Ex. 2. Required the cube root of 10648.

$$\log 10648^{\frac{1}{3}} = \tfrac{1}{3} \times \log 10648 = 1{\cdot}34242.$$

The number answering to this logarithm is 22, which is therefore the root required.

Ex. 3. Given $a^x b^{nx} = c$; to find x.

Taking the log. of each side of the equation,

$$x \log a + nx \log b = \log c;$$

$$\therefore\ x = \frac{\log c}{\log a + n \log b} = \frac{\log c}{\log (ab^n)}.$$

In calculating by logarithms it appears, therefore, that multiplication is effected by addition, division by subtraction, the raising of powers by multiplication, and the extraction of roots by a process of division. The use of the tables may be learnt from the explanations given in the prefaces to such books.

19. Since $\log 10_n = n$,

$$\therefore\ \log (\text{N} \times 10^n) = \log \text{N} + \log 10^n = \log \text{N} + n;$$

from this it follows that if $\log 3{\cdot}754 = {\cdot}57449$, then $\log 375{\cdot}4$ or $3{\cdot}754 \times 10^2 = \log 3{\cdot}754 + \log 10^2 = {\cdot}57449 + 2 = 2{\cdot}57449$. And so on to other cases.

$$\text{Again, } \log \frac{\text{N}}{10^n} = \log \text{N} - \log 10^n = \log \text{N} - n;$$

from this it follows that if $\log 5{\cdot}621 = {\cdot}7498$, then $\log {\cdot}05621$ or $\dfrac{5{\cdot}621}{10^2} = {\cdot}7498 - 2$, or as this is usually written $\bar{2}{\cdot}7498$.

And so on to other cases.

20. To avoid minus characteristics in the logarithm tables of sines and cosines, the radius is taken 10^{10}. If $\sin \text{A}$ be the symbol for the sine of A when the radius of the circle is unity, and Sin A the symbol for the sine of the same arc when the radius is 10^{10}, then by Art. **10.**, we have,

$$\sin \text{A} = \frac{\text{Sin A}}{10^{10}};\ \therefore\ \log \sin \text{A} = \log \text{Sin A} - 10.$$

21. To calculate a from the equation $\sin \text{A} = \dfrac{a}{b}$, &c. (See Art. **11.**)

Here $a = b \sin \text{A}$; then by Art. **16.**,

$$\log a = \log b + \log \sin \text{A}$$
$$= \log b + \log \text{Sin A} - 10 \ldots\ldots (1).$$

And so on to similar cases.

Also in the oblique-angled triangle, $\frac{b}{c} = \frac{\sin \text{B}}{\sin \text{C}}$;

$$\therefore \log b - \log c = \log \text{Sin B} - \log \text{Sin C} \ldots (2);$$

where any three terms being given, the remaining one may be found.

As an illustration of this method of calculation let us take a few examples.

Ex. 1. In the right-angled triangle ABC (see *fig.* Art. **11.**), AC $= 363$, and $\angle$ A $= 63° 25'$; required BC.

$$\text{Here } \sin \text{A} = \frac{\text{BC}}{\text{AC}},\ \therefore \text{BC} = \text{AC} \cdot \sin \text{A};$$

$$\text{whence } \log \text{BC} = \log \text{AC} + \log \text{Sin A} - 10.$$

$$\begin{array}{rcl} \log \text{AC or } 363 & = & 2{\cdot}559907 \\ \log \text{ Sin A} & = & 9{\cdot}951476 \\ -10 & = & -10 \\ \hline \log \text{BC} & = & 2{\cdot}511383 = \log 324{\cdot}6 \\ \therefore \text{BC} & = & 324{\cdot}6 \end{array}$$

Ex. 2. In the triangle ABC (Art. **11.**), given $c = 50$, $\angle$ A $= 74° 14'$, $\angle$ B $= 49° 23'$, and $\therefore \angle$ C $= 56° 23'$; to find b.

Here from eq. (2), Art. **21.**, we have,

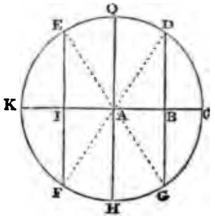
$$\log b = \log c + \log \text{Sin B} - \log \text{Sin C}$$

$$\begin{array}{rcl} \log \text{C or } 50 & = & 1{\cdot}69897 \\ \log \text{ Sin B} & = & 9{\cdot}88029 \\ -\log \text{ Sin C} & = & -9{\cdot}92052 \\ \hline \log b & = & 1{\cdot}65874 = \log 45{\cdot}58 \end{array}$$

$$\therefore \text{AC or } b = 45{\cdot}58.$$

As further exercises the student may solve, by the method of logarithms, all the examples given in Arts. **11. 12.** and **13.**

22. *To trace the signs of sin* A, *cos* A, *&c. as the arc* A *increases from* 0° *to* 360°.



An inspection of this figure will show, 1. When the arc CD is less than 90°, the sine and cosine are both positive. 2. When the arc CE is greater than 90°, the sine, EI, is positive; but the cosine, AI, is negative. 3. When the arc CQF is greater than 180° or 2 right angles, the sine, FI, is negative; and the cosine, AI, is also negative. 4. When the arc CQHG is greater than 270° or 3 right angles, the sine, GB, is negative; but the cosine, AB, is positive. 5. Let 2π be put for the whole circumference of the circle or 360°, and let A be put for the degrees in any arc CD, then if we add 2π to A it will obviously bring us to the same point D in the circle; or generally if we add any number of times 2π to A, it will always bring us to the same point in the circle; and this will be true whether A be taken below the diameter CK or above it, that is, whether A be positive or negative. Hence we have, $\sin A = \sin(n \times 2\pi + A)$, where the sign will be positive or negative according as A is positive or negative, or according as the point D is in the semicircle CQK or CHK. In like manner $\cos A = \cos(n \times 2\pi + A)$, where the sign will be positive or negative according as the point D is in the semicircle HCQ or QKH. 6. When $A = 0$ the sine, DB, evidently becomes 0, and the cosine, AB, becomes AC or 1; when $A = 90°$ the sine DB, becomes QA or 1, and the cosine, AB, becomes 0; when $A = 180°$ the sine again becomes 0, and the cosine becomes -1; when $A = 270°$ the sine becomes -1, and the cosine becomes 0; when $A = 360°$ the sine becomes 0, and the cosine becomes 1. Expressing these results analytically, we have, $\sin 0 = 0$, $\cos 0 = 1$, $\sin\frac{\pi}{2} = 1$, $\cos\frac{\pi}{2} = 0$, $\sin\pi = 0$, $\cos\pi = -1$, $\sin\frac{3\pi}{2} = -1$, $\cos\frac{3\pi}{2} = 0$, $\sin 2\pi = 0$, and

$\cos 2\pi = 1$. It is also evident that if any of these arcs be increased by 2π or any multiple of it, the result will not be altered. 7. If the student thoroughly understands the nature of these results, he will find no difficulty in tracing the signs of any trigonometrical expression which may hereafter occur. For example, let it be required to find the sign of the tangent of an arc A, greater than 90° and less than 180°. In this case, sin A is positive, while cos A is negative,

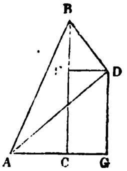
$$\therefore \tan A = \frac{+\sin A}{-\cos A} = \text{a negative quantity.}$$

23. *Trigonometrical formulæ involving two or more angles.*

To prove that,

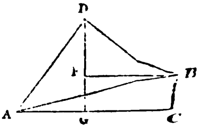
$$\sin(A+B) = \sin A \,.\, \cos B + \cos A \,.\, \sin B \ldots (1)$$
$$\cos(A+B) = \cos A \,.\, \cos B - \sin A \,.\, \sin B \ldots (2)$$
$$\sin(A-B) = \sin A \,.\, \cos B - \cos A \,.\, \sin B \ldots (3)$$
$$\cos(A-B) = \cos A \,.\, \cos B + \sin A \,.\, \sin B \ldots (4)$$

Let $\angle GAD = A$, and $\angle DAB = B$, then $\angle GAB = A + B$. From B, any point in AB, let fall BC and BD perpendiculars on AG and AD; and from D, let fall DG and DF perpendiculars, on AG and BC. Then $CF = GD$, $FD = CG$, and because DF is parallel to AG, $\angle ADF = A$, $\therefore \angle DBF = 90° - \angle BDF = \angle ADF = A$. Now because $\angle BAC = A + B$,



$$\sin(A+B) = \frac{BC}{AB} = \frac{DG+BF}{AB} = \frac{DG}{AB} + \frac{BF}{AB}$$
$$= \frac{DG}{AD} \,.\, \frac{AD}{AB} + \frac{BF}{BD} \,.\, \frac{BD}{AB}$$
$$= \sin A \,.\, \cos B + \cos A \,.\, \sin B.$$

Also
$$\cos(A+B) = \frac{AC}{AB} = \frac{AG-DF}{AB} = \frac{AG}{AB} - \frac{DF}{AB}$$
$$= \frac{AG}{AD} \,.\, \frac{AD}{AB} - \frac{DF}{BD} \,.\, \frac{BD}{AB}$$
$$= \cos A \,.\, \cos B - \sin A \,.\, \sin B.$$



Let $\angle CAD = A$, and $\angle BAD = B$, then $\angle CAB = A - B$. From B, any point in AB, let fall BC and BD perpendiculars on AC and AD; from D let fall DG perpendicular to AC; and from B let fall BF perpendicular to DG. Then $BF = CG$, $BC = FG$, and $\angle BDF = 90^\circ - \angle ADG = \angle GAD = A$. Now because $\angle CAB = A - B$,

$$\sin(A - B) = \frac{BC}{AB} = \frac{DG - DF}{AB} = \frac{DG}{AB} - \frac{DF}{AB}$$

$$= \frac{DG}{AD} \cdot \frac{AD}{AB} - \frac{DF}{DB} \cdot \frac{DB}{AB}$$

$$= \sin A \,.\, \cos B - \cos A \,.\, \sin B.$$

$$\text{Also } \cos(A - B) = \frac{AC}{AB} = \frac{AG + BF}{AB} = \frac{AG}{AB} + \frac{BF}{AB}$$

$$= \frac{AG}{AD} \cdot \frac{AD}{AB} + \frac{BF}{BD} \cdot \frac{BD}{AB}$$

$$= \cos A \,.\, \cos B + \sin A \,.\, \sin B.$$

These four expressions constitute the fundamental formulæ in analytical trigonometry, from which a great variety of useful properties may be readily derived. The values of A and B being perfectly general, we may assign to them any particular values we please.

24. To prove that $\sin 2A = 2 \sin A \,.\, \cos A$.

In formula (1) let $B = A$, then $A + B = 2A$, and

$\therefore \sin 2A = \sin A \,.\, \cos A + \cos A \,.\, \sin A = 2 \sin A \,.\, \cos A$.

25. To prove that $\cos 2A = \cos^2 A - \sin^2 A$, &c.

In formula (2) let $B = A$, then we have,

$$\cos 2A = \cos A \,.\, \cos A - \sin A \,.\, \sin A = \cos^2 A - \sin^2 A.$$

In this expression put first $1 - \cos^2 A$ for $\sin^2 A$, and afterwards $1 - \sin^2 A$ for $\cos^2 A$ (Art. **5.**) then,

$$\cos 2A = \cos^2 A - (1 - \cos^2 A) = 2 \cos^2 A - 1$$
$$\cos 2A = 1 - \sin^2 A - \sin^2 A = 1 - 2 \sin^2 A.$$

26. In the formulæ of the last two articles, let $2A = a$, and

$\therefore A = \frac{a}{2}$, then,

$$\sin a = 2 \sin \frac{a}{2} . \cos \frac{a}{2}$$

$$\cos a = \cos^2 \frac{a}{2} - \sin^2 \frac{a}{2} = 2 \cos^2 \frac{a}{2} - 1.$$

$$\cos a = 1 - 2 \sin^2 \frac{a}{2}.$$

27. Adding and subtracting formulæ (1) and (3), Art. **23.**,

$$\sin(\text{A} + \text{B}) + \sin(\text{A} - \text{B}) = 2 \sin \text{A} . \cos \text{B} \ldots (1)$$
$$\sin(\text{A} + \text{B}) - \sin(\text{A} - \text{B}) = 2 \cos \text{A} . \sin \text{B} \ldots (2)$$

Adding and subtracting formulæ (2) and (4),

$$\cos(\text{A} + \text{B}) + \cos(\text{A} - \text{B}) = 2 \cos \text{A} . \cos \text{B} \ldots (3)$$
$$\cos(\text{A} - \text{B}) - \cos(\text{A} + \text{B}) = 2 \sin \text{A} . \sin \text{B} \ldots (4)$$

28. To find the sine and cosine of an arc in terms of the sine of double the arc.

$$\cos^2 \text{A} + \sin^2 \text{A} = 1$$
$$2 \sin \text{A} . \cos \text{A} = \sin 2 \text{A};$$

∴ by addition and subtraction,

$$\cos^2 \text{A} + 2 \sin \text{A} . \cos \text{A} + \sin^2 \text{A} = 1 + \sin 2 \text{A},$$
$$\cos^2 \text{A} - 2 \sin \text{A} . \cos \text{A} + \sin^2 \text{A} = 1 - \sin 2 \text{A};$$

taking the square root of each side,

$$\cos \text{A} + \sin \text{A} = \pm \sqrt{1 + \sin 2 \text{A}},$$
$$\cos \text{A} - \sin \text{A} = \pm \sqrt{1 - \sin 2 \text{A}}.$$

If A is less than 45°, then it is evident that cos A will be greater than sin A; in this case therefore the two roots will be both positive. By addition and subtraction, and dividing by 2.

$$\cos \text{A} = \tfrac{1}{2}\left\{ \sqrt{1 + \sin 2 \text{A}} + \sqrt{1 - \sin 2 \text{A}} \right\}$$
$$\sin \text{A} = \tfrac{1}{2}\left\{ \sqrt{1 + \sin 2 \text{A}} - \sqrt{1 - \sin 2 \text{A}} \right\}.$$

29. Given the tangents of two angles, to find the tangent of their sum or difference.

$$\tan(\text{A} + \text{B}) = \frac{\sin(\text{A} + \text{B})}{\cos(\text{A} + \text{B})} = \frac{\sin \text{A} . \cos \text{B} + \cos \text{A} . \sin \text{B}}{\cos \text{A} . \cos \text{B} - \sin \text{A} . \sin \text{B}}$$

and dividing numerator and denominator by cos A . cos B,

$$\tan(A+B) = \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{1 - \frac{\sin A}{\cos A} \cdot \frac{\sin B}{\cos B}} = \frac{\tan A + \tan B}{1 - \tan A . \tan B}$$

Similarly it may be shown that, $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A . \tan B}$.

Cor. 1. If $B = A$, $\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$.

Cor. 2. If $B = 45°$, then since $\tan 45° = 1$, (Art. 8.)

$$\tan(A + 45°) = \frac{1 + \tan A}{1 - \tan A}$$

$$= \frac{1 + \frac{\sin A}{\cos A}}{1 - \frac{\sin A}{\cos A}} = \frac{\cos A + \sin A}{\cos A - \sin A}.$$

30. By development and multiplication,

$$\sin(A+B) . \sin(A-B) = \sin^2 A . \cos^2 B - \cos^2 A . \sin^2 B$$
$$= \sin^2 A - \sin^2 B$$

similarly $\cos(A+B) . \cos(A-B) = \cos^2 A - \sin^2 B$.

31. In formulæ (1) and (3) of Art. **27.**, for A put na, and for B put a; then $A + B = na + a = (n+1)a$, and $A - B = na - a = (n-1)a$;

$$\therefore \sin(n+1)a + \sin(n-1)a = 2\sin na . \cos a, \ldots (1)$$
$$\text{and } \cos(n+1)a + \cos(n-1)a = 2\cos na . \cos a \ldots (2).$$

If $n = 1$, from (1),

$$\sin 2a = 2\sin a . \cos a;$$
$$\text{from (2), } \cos 2a + 1 = 2\cos^2 a,$$
$$\therefore \cos 2a = 2\cos^2 a - 1.$$

If $n = 2$, from (1),

$$\sin 3a + \sin a = 2\sin 2a . \cos a = 4\sin a . \cos^2 a$$
$$= 4\sin a(1 - \sin^2 a)$$
$$= 4\sin a - 4\sin^3 a,$$
$$\therefore \sin 3a = 3\sin a - 4\sin^3 a.$$

Similarly from (2), we find,

$$\cos 3\,a = 4 \cos^3 a - 3 \cos a.$$

These last two formulæ give the values of the sine and cosine of the triple arc, in terms of the sine and cosine of the simple arc. By successive substitution the sine and cosine of any multiple arc may be found.

32. In the four formulæ Art. **27.**, let $\text{A} + \text{B} = a$, and $\text{A} - \text{B} = b$, in which case $\text{A} = \frac{a+b}{2}$, and $\text{B} = \frac{a-b}{2}$,

$$\therefore \sin a + \sin b = 2 \sin \frac{a+b}{2} \,.\, \cos \frac{a-b}{2} \ldots (1)$$

$$\sin a - \sin b = 2 \cos \frac{a+b}{2} \,.\, \sin \frac{a-b}{2} \ldots (2)$$

$$\cos a + \cos b = 2 \cos \frac{a+b}{2} \,.\, \cos \frac{a-b}{2} \ldots (3)$$

$$\cos b - \cos a = 2 \sin \frac{a+b}{2} \,.\, \sin \frac{a-b}{2} \ldots (4).$$

33. Dividing (2) by (1) of the preceding Article, observing that $\frac{\text{sine}}{\text{cosine}} = \text{tangent}$,

$$\frac{\sin a - \sin b}{\sin a + \sin b} = \frac{\tan \frac{a-b}{2}}{\tan \frac{a+b}{2}}.$$

Similarly, dividing (4) by (3),

$$\frac{\cos b - \cos a}{\cos a + \cos b} = \tan \frac{a+b}{2} \,.\, \tan \frac{a-b}{2}.$$

As a and b in these formulæ express any angles whatever, we may put A for a, and B for b.

34. *To calculate the sines, cosines, &c. of arcs.*

(1.) Having given the sines and cosines of any two arcs, we can always find the sine and cosine of their sum or difference; for example,

$$\sin 75^\circ = \sin(45^\circ + 30^\circ) = \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{1}{2}\sqrt{3} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

Similarly, $\cos 75^\circ = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$. Also, $\sin 75^\circ = \cos 15^\circ$, and $\cos 75^\circ = \sin 15^\circ$. And $\tan 75^\circ = \dfrac{\sqrt{3}+1}{2\sqrt{2}} \div \dfrac{\sqrt{3}-1}{2\sqrt{2}} = \dfrac{\sqrt{3}+1}{\sqrt{3}-1}$ $= 2 + \sqrt{3}$.

(2.) To find the sine and cosine of 18° and 36°,

$$\cos 54^\circ = \sin 36^\circ, \text{ or } \cos 3 \,.\, 18^\circ = \sin 2 \,.\, 18^\circ.$$

But by Art. **31.**,

$$\cos 3 \,.\, 18^\circ = 4\cos^3 18^\circ - 3\cos 18^\circ$$

and $\sin 2 \,.\, 18^\circ = 2 \sin 18^\circ \cos 18^\circ$, $\therefore$ by equality,

$$4\cos^3 18^\circ - 3\cos 18^\circ = 2\sin 18^\circ \cos 18^\circ,$$

$$\therefore 4\cos^2 18^\circ - 3 = 2\sin 18^\circ \ldots (1).$$

Let $\sin 18^\circ = x$, then $\cos^2 18^\circ = 1 - x^2$, and this equation becomes, $4x^2 + 2x = 1$, whence we find

$$x = \frac{\sqrt{5}-1}{4} = \sin 18^\circ \text{ or } \cos 72^\circ.$$

Substituting this value of $\sin 18^\circ$ in eq. (1),

$$4\cos^2 18 - 3 = \frac{\sqrt{5}-1}{2}, \; \therefore \cos 18^\circ = \frac{\sqrt{5+\sqrt{5}}}{2\sqrt{2}}.$$

Again, $\sin 54^\circ = \cos 36^\circ = \cos 2 \,.\, 18^\circ$

$$= \cos^2 18^\circ - \sin^2 18^\circ = \frac{5+\sqrt{5}}{8} - \frac{3-\sqrt{5}}{8} = \frac{1+\sqrt{5}}{4}.$$

$$\cos^2 54^\circ = 1 - \sin^2 54^\circ = 1 - \frac{3+\sqrt{5}}{8} = \frac{5-\sqrt{5}}{8},$$

$$\therefore \cos 54^\circ \text{ or } \sin 36^\circ = \frac{\sqrt{10-2\sqrt{5}}}{4}.$$

(3.) Having given the sine of any arc, we can always find the sine and cosine of half the arc, by the two formulæ of Art. **28.** Thus, as $\sin 30^\circ = \frac{1}{2}$, we have.

$$\sin 15^\circ = \tfrac{1}{2}\left\{\sqrt{1+\tfrac{1}{2}} - \sqrt{1-\tfrac{1}{2}}\right\} = \tfrac{1}{2}\left(\sqrt{\tfrac{3}{2}} - \sqrt{\tfrac{1}{2}}\right).$$

We may then successively proceed to determine the sines of $\frac{15^\circ}{2}, \frac{15^\circ}{2^2}, \frac{15^\circ}{2^3} \ldots \frac{15^\circ}{2^{10}}$.

In this way we find $\sin \frac{15^\circ}{2^{10}}$ or $\sin \frac{225}{256} . 1' = \cdot000255625$. Now in very small arcs like this, the sine varies (practically) as the arc itself;

$$\therefore \sin 1' = \frac{256}{225} \times \cdot000255625 = \cdot00029088820.$$

The cosine and tangent of 1′ can now be found,

$$\cos 1' = \sqrt{1 - \sin^2 1'} = 9999999577,$$

$$\text{and } \tan 1' = \frac{\sin 1'}{\cos 1'} = \cdot00029088820.$$

Here it will be observed that the values of the sine and tangent of 1′ agree as far as the first ten decimal figures.

Now (Art. **73.** Geo.) the arc of a circle is greater than the sine and less than the tangent; but as the values of the sine and tangent of an arc of 1′, agree as far as the figures here given, it follows that,

The arc of 1′ to rad. unity = ·00029088820, &c.

Since there are 21600′ in the whole circle, the circumference of the circle will be 21600 times this quantity, or 6·283185 &c.; and this result must obviously be correct, at least, as far as the sixth decimal place. But this quantity is the circumference of a circle, whose radius is 1 or diameter 2, therefore the circumference of a circle, whose diameter is unity, will be ½ of 6·283185 &c. = 3·14159 &c. This is the value of the symbol π. In practical calculations this number is usually taken 3·1416.

Hence we have, by Art. **75.** Geo., Cor.,

Circum. circle whose diam. is $D = D \times 3{\cdot}1416$,
and Area „ „ $= D^2 \times {\cdot}7854$.

(4.) To find the sine and cosine of 2′, 3′, 4′, &c.

In the formulæ (1) and (2), of Art. **31.**, put $a = 1'$, and

$n = 1, 2, 3,$ &c., and the resulting expressions will enable us to calculate the sines and cosines of all angles for every minute of a degree.

For further details on the construction of tables, the student may consult *Davies's edition of Hutton*, or *Hall's Trigonometry.*

Exercises.

1. Prove that $\cos 2A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$;

$$\tan A = \sqrt{\frac{1 - \cos 2A}{1 + \cos 2A}};$$

$$\tan A = \cot A - 2 \cot 2A;$$

$$\sin (30° + A) = \cos A - \sin (30° - A).$$

2. Show that $\cos 11° \; 15' = \frac{1}{2}\sqrt{2 + \sqrt{2 + \sqrt{2}}}$; and $\tan 22° \; 30' = \sqrt{2} - 1$.

3. If $A + B + C = 180°$, show that,

$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$$

4. $(\cot A \,.\, \cos A)^2 = \cot^2 A - \cos^2 A.$

5. $\sin A = \dfrac{\tan A}{\sqrt{1 + \tan^2 A}}$; $\tan (45° + \frac{1}{2} A) = \dfrac{\cos A}{1 - \sin A}.$

6. $\tan A = \dfrac{\sin 2A}{1 + \cos 2A} = \dfrac{2 \cot \frac{1}{2} A}{(\cot \frac{1}{2} A)^2 - 1}.$

7. $\sec A = 1 + \tan A \,.\, \tan \frac{1}{2} A.$

8. $(\sec 2A)^2 = \dfrac{2 \sec 2A}{1 + \sec 2A}$; $\cot A = \dfrac{\sin 2A}{1 - \cos 2A}.$

9. $\sin 12° = \frac{1}{8} \{ \sqrt{5} - 1 - \sqrt{6 (5 + \sqrt{5})} \}.$

10. Determine x in the following equations:—

$\tan 2x = 3 \tan x.$	*Ans.* $x = 30°.$
$\tan x = \operatorname{cosec} 2x.$	*Ans.* $x = 45°.$
$2 \cos 2x = 2 \sin x + 1.$	*Ans.* $x = 18°$, &c.
$\sin (x - a) = \cos (x + a).$	*Ans.* $x = 45°.$

35. *To find the cosine of the angle of a triangle in terms of the sides.*

Now in fig. 2., Geometry, Art. 50.,

$$\text{AB}^2 = \text{BC}^2 + \text{AC}^2 - 2\,\text{BC}\,.\,\text{DC};$$

in fig. 1., Geometry, Art. 49.,

$$\text{AB}^2 = \text{BC}^2 + \text{AC}^2 + 2\,\text{BC}\,.\,\text{DC}.$$

But in fig. 2., $\frac{\text{DC}}{\text{AC}} = \cos \text{C}$, or $\text{DC} = \text{AC}\,.\,\cos \text{C}$;

and in fig. 1., $\frac{\text{DC}}{\text{AC}} = \cos \text{ACD} = -\cos \text{ACB}$ or $-\cos \text{C}$;

$\therefore\ \text{DC} = -\text{AC}\,.\,\cos \text{C}$. Therefore in both cases, by substitution,

$$\text{AB}^2 = \text{BC}^2 + \text{AC}^2 - 2\,\text{BC}\,.\,\text{AC}\,.\,\cos \text{C},$$

$$\therefore\ \cos \text{C} = \frac{\text{BC}^2 + \text{AC}^2 - \text{AB}^2}{2\,\text{BC}\,.\,\text{AC}};$$

or adopting the notation of Art. 11.,

$$\cos \text{C} = \frac{a^2 + b^2 - c^2}{2\,ab},$$

and similarly $\cos \text{A} = \frac{b^2 + c^2 - a^2}{2\,bc}$; and so on.

In order to adapt this formula to logarithmic computation, we have by adding 1 to each side of the equality,

$$1 + \cos \text{A} = \frac{b^2 + 2\,bc + c^2 - a^2}{2\,bc} = \frac{(b+c)^2 - a^2}{2\,bc}$$

$$= \frac{(b+c+a)\,(b+c-a)}{2\,bc}.$$

But by Art. 26., $1 + \cos \text{A} = 2 \cos^2 \frac{\text{A}}{2}$; and if s be put for $\frac{a+b+c}{2}$, $\therefore\ 2\,\text{s} = a+b+c$, and $2\,(\text{s}-a) = b+c-a$;

$$\therefore\ 2\cos^2 \frac{\text{A}}{2} = \frac{2\,\text{s} \times 2\,(\text{s}-a)}{2\,bc}$$

$$\therefore\quad \cos \frac{\text{A}}{2} = \sqrt{\frac{\text{s}\,(\text{s}-a)}{b\,c}} \ldots (1)$$

Or adapting the formula to radius r, (see Art **10.**)

$$\cos\frac{A}{2} = r\sqrt{\frac{s(s-a)}{bc}}; \text{ or in logarithms,}$$

$$\log\cos\frac{A}{2} = 10 + \tfrac{1}{2}\{\log s + \log(s-a) - \log b - \log c\} \ldots (2).$$

Similarly we find,

$$1 - \cos A = \frac{a^2 - (b^2 - 2bc + c^2)}{2bc} = \frac{2(s-b)(s-c)}{bc};$$

$$\text{but } 1 - \cos A = 2\sin^2\frac{A}{2}, \text{ (Art. } \mathbf{26.}\text{)}$$

$$\therefore \sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \ldots (3).$$

Dividing (3) by (1),

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \ldots (4).$$

Multiplying (3) and (1), we have,

$$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} \ldots (5).$$

These three last formulæ may be expressed in logarithms, after the same manner as formula (1).

By any of the formulæ, (1), (3), (4), and (5), *the angles of a triangle may be computed, when the sides are given.* If, however, the angle required be near 90°, the last formula should not be used, because the sines of arcs near 90° differ but little from each other.

Examples.

Given the three sides of a triangle to find an angle.

Ex. 1. Let $a = 7$, $b = 9$, and $c = 8$; required $\angle A$.

Here by eq. (1), $s = \frac{1}{2}(7 + 9 + 8) = 12$,

$$\cos\frac{A}{2} = \sqrt{\frac{12(12-7)}{9 \times 8}} = \cdot 91287;$$

$$\therefore \frac{\text{A}}{2} = 24° \; 6'; \text{ and } \text{A} = 48° \; 12'.$$

Or by logarithms, eq. (2),

$$\log\cos\frac{\text{A}}{2} = 10 + \tfrac{1}{2}\{\log 12 + \log 5 - \log 9 - \log 8\} = 9{\cdot}9604;$$

$$\therefore \frac{\text{A}}{2} = 24° \; 6', \text{ and } \text{A} = 48° \; 12'.$$

2. Given the three sides of a triangle 81, 135, and 108; to find the angle opposite to the least side.

To solve this question by eq. (3). Putting a for 81, the opposite angle, which we have to find, will be A, and then the two remaining sides will be b and c; and $\text{s} = \frac{1}{2}(81 + 135 + 108) = 162$.

$$\therefore \sin\frac{\text{A}}{2} = \sqrt{\frac{(162-135)\,(162-108)}{135 \times 108}} = \sqrt{\cdot 1}.$$

$$\therefore \log\sin\frac{\text{A}}{2} = 10 + \tfrac{1}{2}\log{\cdot 1} = 9{\cdot}5;\; \therefore \text{A} = 36° \; 52'.$$

3. Required the remaining angles of the triangle in the last example. (See eq. (1), Art. **11.**) *Ans.* 90° and 53° 7′.

4. The three sides of a triangle are 53, 53, and 92·36; required the angle opposite to the greatest side.

Ans. 121° 14′.

5. The three sides of a triangle are 112·65, 120, 112; required the angle opposite to the first named side.

Ans. 57° 58′.

36. *To find the area of a triangle in terms of the sides.*

Let a, b, c be the given sides; then, Art. **12.**,

$$\text{Area } \Delta = \tfrac{1}{2}\, b\, c \times \sin \text{A};$$

substituting the value of sin A, given in eq. (5), Art. **35.**,

$$\text{Area } \Delta = \sqrt{\text{s}\,(\text{s}-a)\,(\text{s}-b)\,(\text{s}-c)}$$

Ex. 1. What is the area of a triangle, whose three sides are 25·69, 49, and 50·25?

Here $s = \frac{1}{2}(25{\cdot}69 + 49 + 50{\cdot}25) = 62{\cdot}47$, $s - a = 62{\cdot}47 - 25{\cdot}69 = 36{\cdot}78$, $s - b = 13{\cdot}47$, and $s - c = 12{\cdot}22$;

$$\therefore \text{ Area } \Delta = \sqrt{62{\cdot}47 \times 36{\cdot}78 \times 13{\cdot}47 \times 12{\cdot}22} = 614{\cdot}9.$$

Or by logarithms,

$$\log \text{Area } \Delta = \tfrac{1}{2}(\log 62{\cdot}47 + \log 36{\cdot}78 + \log 13{\cdot}47 + \log 12{\cdot}22).$$

2. What is the area of a triangle, whose three sides are 20, 30, and 40? *Ans.* 290·47.

37. *Given any two sides of a triangle and the included angle, to find the other parts.*

By eq. (1), Art. **11.**, $\frac{b}{c} = \frac{\sin B}{\sin C}$; adding and subtracting 1,

$$\frac{b-c}{c} = \frac{\sin B - \sin C}{\sin C}; \quad \frac{b+c}{c} = \frac{\sin B + \sin C}{\sin C};$$

by division and Art. **33.**, we have,

$$\frac{b-c}{b+c} = \frac{\sin B - \sin C}{\sin B + \sin C} = \tan \frac{B-C}{2} \div \tan \frac{B+C}{2};$$

but $B + C = 180° - A$, and $\tan \frac{B+C}{2} = \tan \frac{180° - A}{2} = \cot \frac{A}{2}$;

$$\therefore \tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}.$$

$$\therefore \log \tan \frac{B-C}{2} = \log (b-c) - \log (b+c) + \log \cot \frac{A}{2}.$$

The angle $\frac{B-C}{2}$ being determined from this formula, and $B + C$ being known, we can readily find $\angle B$ and $\angle C$. The remaining side of the triangle may be found by Art. **11.**

Ex. 1. Given $b = 345$, $c = 174{\cdot}07$, and $\angle A = 37° \, 20'$; to find $\angle B$ and $\angle C$.

Here, $b - c = 170{\cdot}93$, $b + c = 519{\cdot}07$, and $\frac{A}{2} = 18° \, 40'$; therefore by the formula, we have,

$$\begin{array}{rcr} \log 170{\cdot}93 & = & 2{\cdot}2328183 \\ -\log 519{\cdot}07 & = & -\ 2{\cdot}7152259 \\ \cot 18^\circ\ 40' & = & \underline{10{\cdot}4712979} \end{array}$$

$$\log\tan\frac{\text{B}-\text{C}}{2} = \quad 9{\cdot}9888903 = \log\tan 44^\circ\ 16'.$$

$$\therefore\ \frac{\text{B}-\text{C}}{2} = 44^\circ\ 16'\text{; but } \frac{\text{B}+\text{C}}{2} = \frac{180^\circ - 37^\circ\ 20'}{2} = 71^\circ\ 20'\text{;}$$

adding these equations together, we find, $\text{B} = 115^\circ\ 36'$, and by subtracting we find $\angle\,\text{C} = 27^\circ\ 4'$. The remaining side will be found, by Art. **11.**, to be 232.

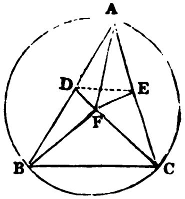
2. The two sides of a triangle are 46·18, and 26·5, and the included angle is 29° 23′; required the remaining angles and side. *Ans.* 121° 14′, 29° 23′, and the side 53.

38. *To find* R *and r, the radii of the circles described in and about a triangle whose sides are given.*

By Exercise 1, p. 65., area $\Delta = r\,.\,\text{s}$;

$$\therefore\ \text{Art. } \mathbf{36.},\ r\,.\,\text{s} = \sqrt{\text{s}\,(\text{s}-a)\,(\text{s}-b)\,(\text{s}-c)},\ \text{or area }\Delta\text{;}$$

$$\therefore\ r = \frac{\sqrt{(\text{s}-a)\,(\text{s}-b)\,(\text{s}-c)}}{\text{s}},\ \text{or}\,\frac{\text{area }\Delta}{\text{s}}\ \ldots(1).$$



Let F be the centre of the circumscribed circle ABC, then $\text{FC} = \text{FA} = \text{FB} = \text{R}$. Draw FE perpendicular to AC, then the triangles CFE and AFE will be identical, and ∴ $\text{CE} = \text{AE}$, and $\angle\,\text{CFE} = \frac{1}{2}\angle\,\text{CFA}$;

but, Geo. Art. **60.**, $\angle\,\text{CBA} = \frac{1}{2}\angle\,\text{CFA}$; ∴ $\angle\,\text{CFE} = \angle\,\text{CBA}$;

and $\sin\text{CFE}$ or $\sin\text{B} = \dfrac{\text{CE}}{\text{CF}} = \dfrac{\frac{1}{2}b}{\text{R}}$; ∴ by eq. (5), Art. **35.**

$$\frac{\frac{1}{2}b}{\text{R}} = \frac{2}{ac}\sqrt{\text{s}\,(\text{s}-a)\,(\text{s}-b)\,(\text{s}-c)},$$

$$\therefore\ \text{R} = \frac{abc}{4\ \text{area }\Delta}\ldots(2).$$

Multiplying (1) and (2), $\text{R}r = \frac{abc}{4\,\text{s}} \ldots (3)$.

Ex. 1. Required r and R, in a triangle whose sides are 6, 8, and 10.

Here, area $\Delta = 24$, and $\text{s} = 12$; $\therefore$ by eq. (1), $r = \frac{24}{12} = 2$; and by eq. (2), $\text{R} = \frac{6 \times 8 \times 10}{4 \times 24} = 5$.

39. *To find the area of a regular polygon of n sides.* (See fig. p. 65.)

Here $\angle\,\text{AOE} = \frac{360^\circ}{n}$, $\therefore$ $\angle\,\text{AOE}' = \frac{180^\circ}{n}$; by Art. **71**. Geo.

$$\text{Area polygon} = n \times \tfrac{1}{2}\text{AE} \times \text{OE}';$$

but $\frac{\text{AE}'}{\text{OE}'} = \tan \text{AOE}'$ or $\tan \frac{180^\circ}{n}$; hence $\text{OE}' = \frac{1}{2}\,\text{AE} \,.\, \cot \frac{180^\circ}{n}$;

$$\therefore \text{ Area polygon} = \frac{n}{4} \,.\, \text{AE}^2 \,.\, \cot \frac{180^\circ}{n}.$$

Cor. 1. If r be put for OE′, the radius of the inscribed circle, then

$$r = \tfrac{1}{2}\text{AE} \,.\, \cot \frac{180^\circ}{n}.$$

Cor. 2. Let R be put for OA, the radius of the circumscribed circle, then we have,

$$\frac{\text{AE}'}{\text{OA}} = \sin \frac{180^\circ}{n}; \; \therefore \; \text{R} = \tfrac{1}{2}\,\text{AE} \div \sin \frac{180^\circ}{n}.$$

Cor. 3. $\frac{\text{OE}'}{\text{OA}} = \cos \text{AOE}'$; $\therefore$ $\frac{r}{\text{R}} = \cos \frac{180^\circ}{n}$.

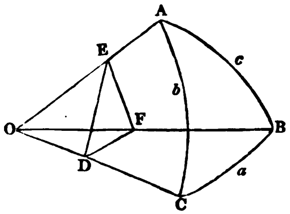
Ex. 1. What is the area of a pentagon whose side is 5?

Here, $\text{AE} = 5$, and $n = 5$, $\therefore$ Area $= \frac{5}{4} \times 5^2 \times \cot.\ 36^\circ =$ 43·01.

2. What is the area of a hexagon whose side is 10?

Ans. 259·8.

To determine the fundamental equations in Spherical Trigonometry. *



40. Let ABC be a spherical triangle, formed by three planes passing through O, the centre of the sphere. Take $OD = 1$, and in the planes AOC and BOC, draw DE and DF each perpendicular to the edge OC, and join EF. Because $\angle$ FDE is the measure of the inclination of the planes AOC and BOC, it will also be the measure of the spherical angle C. Moreover the number of degrees in the side CB or a will measure the $\angle$ COB, and so on.

From the triangles EDF and EOF we have by Art. **35.**,

$$EF^2 = DF^2 + DE^2 - 2DF \, . \, DE \, . \cos FDE$$
$$EF^2 = OF^2 + OE^2 - 2OF \, . \, OE \, . \cos FOE;$$

by subtraction, observing that $OD = 1$, and that ODF and ODE are right-angled triangles,

$$OF \, . \, OE \, . \cos FOE = 1 + DF \, . \, DE \, . \cos FDE,$$

$$\therefore \cos FOE = \frac{1}{OF} \, . \, \frac{1}{OE} + \frac{DF}{OF} \, . \, \frac{DE}{OE} \, . \cos FDE.$$

$$\therefore \cos c = \cos a \, . \cos b + \sin a \, . \sin b \, . \cos C \; \ldots \; (1).$$

$$\therefore \cos C = \frac{\cos c - \cos a \, . \cos b}{\sin a \, . \sin b} \; \ldots \; (2).$$

As this equation is not restricted to any particular sides, we may put B for A, b for a, and a for b, and so on; so that in fact this equation represents three.

To find sin C, we have,

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* The student may pass over this section until he has read the "*Problems on Heights* and Distances."

$$\sin C = \sqrt{1 - \cos^2 C} = \sqrt{(1 + \cos C)\,(1 - \cos C)}\,;$$

adding 1 to each side of eq. (2), and reducing to a common denominator,

$$1 + \cos C = \frac{\cos c - (\cos a \,.\, \cos b - \sin a \,.\, \sin b)}{\sin a \,.\, \sin b}$$

$$= \frac{\cos c - \cos (a + b)}{\sin a \,.\, \sin b} = \frac{2 \sin \dfrac{a+b+c}{2} \,.\, \sin \dfrac{a+b-c}{2}}{\sin a \,.\, \sin b}.$$

$$= \frac{2 \sin S \,.\, \sin (S - c)}{\sin a \,.\, \sin b}. \quad \text{(See Arts. 32 and 35.)}$$

Similarly we find,

$$1 - \cos C = \frac{2 \sin (S - b) \,.\, \sin (S - a)}{\sin a \,.\, \sin b}\,; \quad \therefore \sin C =$$

$$\frac{2}{\sin a \,.\, \sin b} \sqrt{\sin S \,.\, \sin (S - a) \,.\, \sin (S - b) \,.\, \sin (S - c)}.$$

Dividing by sin c, and putting M for the *symmetrical expression* on the right hand, we have, $\dfrac{\sin C}{\sin c} = M$. Now if C and c be changed into B and b, or into A and a, the quantity M will not at all be altered.

$$\therefore \frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c},$$

$$\text{and hence also } \frac{\sin A}{\sin B} = \frac{\sin a}{\sin b}, \text{ \&c.}, \ldots (3).$$

This result shows that *in every spherical triangle, the sines of the angles are in the ratio of the sines of the opposite sides.*

From eq. (1), changing C into A, c into a, and a into c,

$$\cos a = \cos c \,.\, \cos b + \sin c \,.\, \sin b \,.\, \cos A\,;$$

substituting in this equation the value of cos c given in (1), and reducing,

$$\cos a \,.\, \sin b = \cos b \,.\, \sin a \,.\, \cos C + \sin c \,.\, \cos A \ldots (4).$$

Dividing this equation by sin a; substituting $\frac{\sin C}{\sin A}$ for $\frac{\sin c}{\sin a}$; and reducing, we find,

$$\cot a \,.\, \sin b = \cos b \,.\, \cos C + \cot A \,.\, \sin C \;\ldots\; (5).$$

Dividing eq. (4) by sin a; substituting by eq. (3); and multiplying by sin A, we find,

$$\cos a \,.\, \sin B = \cos b \,.\, \cos C \,.\, \sin A + \sin C \,.\, \cos A,$$

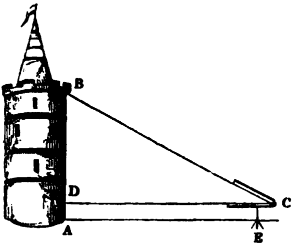
similarly, $\cos b \,.\, \sin A = \cos a \,.\, \cos C \,.\, \sin B + \sin C \,.\, \cos B$, eliminating cos b from these two equations, we finally have,

$$\cos a = \frac{\cos A + \cos B \,.\, \cos C}{\sin B \,.\, \sin C} \;\ldots\; (6).$$

The equations (1), (3), (5), and (6), constitute the fundamental theorems of spherical trigonometry, and serve for the solution of *all* triangles when the letters A, B, &c., are changed one into the other. By making the angles, in these formulæ, right angles, we readily obtain six equalities, which are known by the name of Napier's Analogies.

PROBLEMS IN HEIGHTS AND DISTANCES.

1. From the foot of a tower AB, a surveyor measured AE = 100 ft. on a level plane. He then took the angle DCB = 47° 30′, which the top of the tower formed with the horizontal line CD. Required the height AB of the tower, the height of the instrument EC or AD being 5 ft.



Here, after the method of Ex. 10. Art. 11., we find DB = 109·13, and adding the height of the instrument, AB = 114·13 ft.

2. Required the same as in Problem 4. page 15, when AD = 100 yds., ∠CDB = 46°, and ∠CAB = 31°.

First to find CD. In the triangle ADC we have,

$$\angle \text{ACD} = 46^\circ - 31^\circ = 15^\circ, \text{ and } \frac{\text{CD}}{\text{AD}} = \frac{\sin \text{CAD}}{\sin \text{ACD}},$$

$$\therefore \text{CD} = \frac{\sin \text{CAD}}{\sin \text{ACD}} \times \text{AD}.$$

$$\begin{aligned} \therefore \text{ by logarithms, } \quad \log \sin 31^\circ &= 9{\cdot}7118393 \\ -\log \sin 15^\circ &= 9{\cdot}4129962 \\ \log 100 &= 2 \\ \hline \log \text{CD} &= 2{\cdot}2988431 \end{aligned}$$

To find CB in the triangle BDC. Here $\frac{\text{CB}}{\text{CD}} = \sin \text{CDB}$,

$\therefore$ CB = sin CDB × CD; hence by logarithms,

$$\begin{aligned} \log \sin 46^\circ &= 9{\cdot}8569341 \\ \log \text{CD} &= 2{\cdot}2988431 \\ &\quad -10 \\ \hline \log \text{CB} &= 2{\cdot}1557772 = \log 143{\cdot}14; \end{aligned}$$

$$\therefore \text{CB} = 143{\cdot}14 \text{ yds.}$$

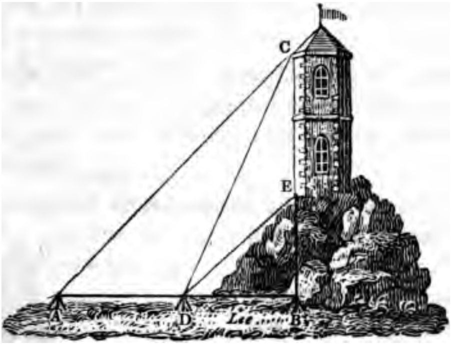
Obs. This distance may at once be found from the equation $\text{CB} = \frac{\sin \text{CDB} \cdot \sin \text{CAD} \cdot \text{AD}}{\sin \text{ACD}}$, which is obtained by substituting in the second equation, the value of CD given in the first.

3. Two observers, 880 yards from each other, in the same vertical plane with a cloud and on the same side of it, find at the same moment its elevation to be 35° and 64°: required the perpendicular height of the cloud. *Ans.* 935·75 *yds.*

4. Wanting to find the breadth of a river at AB (see fig. 1. p. 24.), I measured AC = 200 yds., and then found ∠BAC = 73° 15′, and ∠BCA = 68° 2′: required AB.

Ans. 296·5 *yds.*

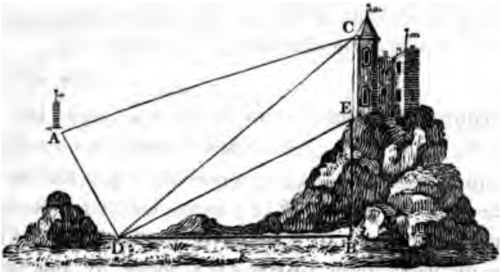
5. Wanting to find the height of a tower CE standing on a hill; at A I took the angle of elevation $CAB = 33^\circ\ 45'$; I



then measured $AD = 100$ yds. in a direct line towards the tower; at D I found the $\angle CDB = 51^\circ$, and the $\angle EDB = 40^\circ$: required the height of the tower.

Here in the triangle ADC, we have given $AD = 100$, $\angle CAD = 33^\circ\ 45'$, and $\angle ACD = \angle CDB - \angle CAD = 17^\circ\ 15'$; hence we find CD. Then in the triangle CDE, we have given CD, $\angle CDE = \angle CDB - \angle EDB = 11^\circ$, and $\angle CED = \angle EDB + \angle EBD = 130^\circ$; hence we find $CE = 46{\cdot}66$ yds.

6. Wanting to find the height of a tower CE, I measured

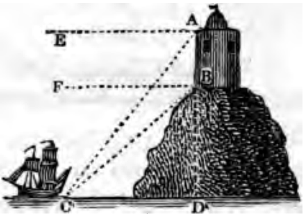


the distance, $AD = 52$ yds., between two stations A and D not in a direct line with the tower; at A I found the $\angle CAD$

$= 64^\circ\ 30'$; at D the $\angle ADC = 72^\circ\ 10'$, $\angle CDB = 58^\circ$, and $\angle EDB = 33^\circ$: required the height of the tower CE, and the height of the hill BE above the level line DB, drawn from the station D. *Ans.* $CE = 34{\cdot}46$, *and* $BE = 23{\cdot}5$ *yds.*

7. What is the perpendicular height of a hill whose angle of elevation, taken at the bottom of it, was 58°, and 100 yds. directly farther off on a level plane, the angle was 32°? *Ans.* 102·5 *yds.*

8. From the top of a lighthouse AB, 85 ft. high, the angle of depression EAC of a ship was 3° 38′; and at the bottom B of the lighthouse, the angle of depression FBC was 2° 43′; required the horizontal distance CD of the vessel, and the height DB of the promontory.



Ans. $CD = 5296$ *ft.*, *and* $DB = 251$ *ft.*

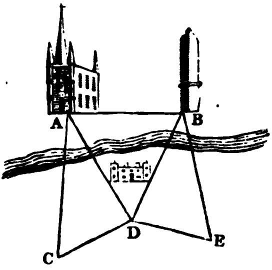
Here in the triangle ABC, $\angle ACD = \angle EAC = 3^\circ\ 38'$, and $\angle BCD = \angle FBC = 2^\circ\ 43'$, $\therefore\ \angle ACB = 3^\circ\ 38' - 2^\circ\ 43' = 55'$; $\angle CAB = 90^\circ - \angle EAC = 86^\circ\ 22'$; and $AB = 85$; hence we find CB. And so on.

9. From the top of a ship's mast 80 ft. above the water, the angle of depression of the hull of another ship was found to be 20°: required the distance between the ships. *Ans.* 219·79 *ft.*

10. Required the same as in Problem 5., page 16., when $BD = 40$ ft., $DA = 60$ ft., $\angle CDB = 41^\circ$, and $\angle CAB = 23^\circ\ 45'$.

Here in the triangle ADC, $AD = 60$, $\angle A = 23^\circ\ 45'$, and $\angle ACD = 41^\circ - 23^\circ\ 45' = 17^\circ\ 15'$; hence we find $CD = 81{\cdot}488$. Then in the triangle DCB we have, $BD = 40$, $CD = 81{\cdot}488$, and the included angle $CDB = 41^\circ$; hence by the formula Art. **37.**, we find $CB = 57{\cdot}6$ ft.

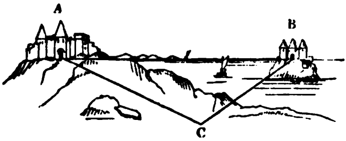
11. Two objects A and B, whose distance from each other



is required, could only be seen at the same time from a station D. I measured the two base lines DC = 200 yds., and DE = 200 yds.; I then took the angles ADC = 89°, ADB = 72° 30′, BDE = 54° 30′, BED = 88° 30′, and DCA = 50° 30′: required AB.

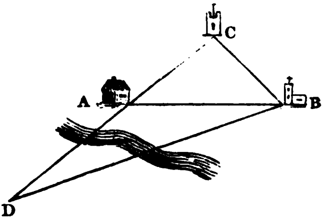
From the triangle ACD, we find, AD = 237·6. From the triangle DBE, we find BD = 332·2. And from the triangle ADB, Art. 37., we find AB = 345·5 yds.

12. Required the same in Exercise 14., page 29., CD = 400 yds., ∠CDA = 53° 20′, ∠ADB = 45° 15′, ∠DCB = 58° 20′, and ∠BCA = 37°. *Ans.* AB = 635 *yds.*



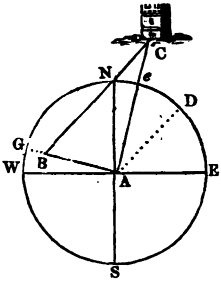
13. Wanting to know the distance between two towers A and B; at a station C I took the angle ACB = 55° 40′; I then measured CA = 7·35 chains, and CB = 8·4 chains: required AB. *Ans.* 7·4 *chains.*

14. The distances of three objects, A, B, and C from each other were known, *viz.* AB = 6 miles, BC = 3·6, and AC = 4. At a station D, in a line with A and C, I found the ∠D = 17° 47′ 20″: required AD.



In the triangle ABC the three sides are given; hence by Art. 35., we find ∠CAB = 35° 34′ 38″. Then in the triangle DAB we have given AB = 6, ∠D = 17° 47′ 20″, and ∠ABD = ∠CAB − ∠D = 17° 47′ 18″; hence we find AD = 6 miles nearly.

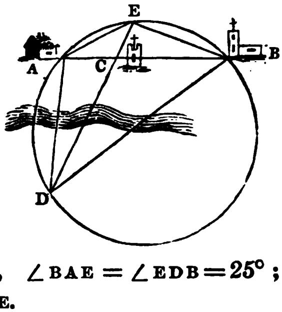
15. From a ship at A I observed a point of land C to bear N.N.E., and after sailing 12 miles in a W.N.W. direction, I found the point of land bore N.E. b. E: required my distance from C at each point of observation.



Construction. Mark off the N.N.E. point of the compass, that is, take $Ne = 22^\circ\ 30'$; join Ae and produce it; set off the W.N.W. point of the compass, that is, take $NG = 67^\circ\ 30'$, and draw AG; on this line take $AB = 12$; set off the N.E.b.E. point, that is, take $ND = 56^\circ\ 15'$; join AD, and from B draw BC parallel to it; then the units in AC and BC will be units of miles in the distances required.

By calculation. In the triangle ABC we have given, $AB = 12$, $\angle BAC = NG + Ne = 67^\circ\ 30' + 22^\circ\ 30' = 90^\circ$, and $\angle BCA = \angle CAD = ND - Ne = 56^\circ\ 15' - 22^\circ\ 30' = 33^\circ\ 45'$; hence we find $BC = 21{\cdot}6$ miles, and $AC = 17{\cdot}9$ miles.

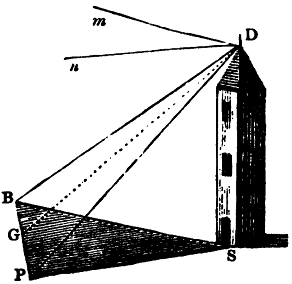
16. Three objects, A, C, and B, in the same straight line, were at a known distance from each other, *viz.* $AC = 1{\cdot}813$, and $CB = 4{\cdot}187$ miles; at a station D I took the angles $ADC = 19^\circ$, and $CDB = 25^\circ$: required the distances DA and DB.



See the construction to Problem 3., page 57. In the triangle ABE, we have given $AB = AC + CB = 6$, $\angle ABE = \angle ADE = 19^\circ$, $\angle BAE = \angle EDB = 25^\circ$; hence we find the sides AE and BE.

In the triangle ACE, we have given AE, AC, and the included angle CAE; hence by Art. **37.**, we find the $\angle C$.

In the triangle ADC, we have given, AC, $\angle ADC$, and $\angle C$; hence we find $DA = 4{\cdot}73$ miles. Similarly in the triangle *DCB, we find* $DB = 8{\cdot}42$ miles.



17. Wishing to find the height of the tower SD, I took a base line PB in the same horizontal plane with the bottom of the tower, and placed stations at P, G, and B, making $PG = GB = a$. At P I found the angle of elevation $= \angle P$; at G the angle of elevation $= \angle G$; and at B the angle of elevation $= \angle B$: required an expression for the height of the tower.

Let $x = SD$; then $SP = x \cot P$, $SG = x \cot G$, and $SB = x \cot B$; therefore by Art. **51.** Geo.,

$$x^2 \cot^2 P + x^2 \cot^2 B = 2a^2 + 2x^2 \cot^2 G;$$

$$\therefore\ x = \sqrt{\frac{2a^2}{\cot^2 P + \cot^2 B - 2\cot^2 G}}.$$

18. At the top of a tower a feet high, the angles of depression of the top and bottom of an upright column, standing on the same level plane as the tower, were found to be 30° and 60°; required the height of the column. *Ans.* $\frac{2a}{3}$.

19. From the top of the Peak of Teneriffe, h miles high above the level of the sea, the visible horizon appeared depressed A°; required the radius of the earth.

Ans. $h \,.\, \cot \frac{A}{2} \,.\, \cot A.$

MENSURATION OF SURFACES.

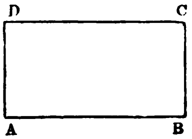
1. Problem. To find the area of a square, rectangle, or any parallelogram.

Rule. Multiply the base by the perpendicular height or breadth, and the product will be the area. (See Geo. pages 31 and 34.)

Note. The dimensions of artificers' work are usually taken in feet and inches. Land is measured by the chain, which is 66 feet long, and contains 100 links. An acre contains 10 sq. chains, or 100,000 sq. links. The content of land is also expressed in roods and perches: 40 perches make 1 rood, and 4 roods make 1 acre.

Examples.

1. Required the area of the rectangle ABCD, whose base AB is 14·80 chains or 1480 links, and breadth BC is 5·45 chains or 545 links.

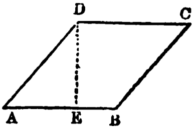


Links.
1480
545
7400
5920
7400
100000)8·06600 acres.

ac. 8·066
4
r. 0·264
40
p. 10·560
Area = 8 *ac.* 0 *r.* 10·56 *p.*

2. Required the area of a parallelogram ABCD, whose *base* AB *is* 3 feet 4 inches, and perpendicular height DE is *2 feet 8 inches.*

3 ft. 4′
2 8
6 8
2 2 8
Area = 8 ft. 10′ 8″



Here, as an inch is the 12th of a foot, we calculate by the scale of twelfths; observing that units multiplied by units give units, units multiplied by twelfths give twelfths, and twelfths multiplied by twelfths give 144ths, or 1 sq. inch. The 12ths are written with an accent over them, and the 144ths with two accents.

3. A door is 7 ft. 3 in. long, and 3 ft. 6 in. broad; required its area, and cost at 2*s.* 3*d.* per sq. foot.

By decimals.
7·25 ft
3·5 ft.
3625
2175
Area 25·375 sq. ft.
2
d. | 50·750
3 = ¼ | 6·3437
57·0937 shillings.
12
1·1244 pence.

Ans. £2 17*s.* 1*d.*

By duodecimals.
7 ft. 3′
3 6
21 9
3 7 6
Area 25 ft. 4′ 6″
s. *d.*
2 3
5
11 3
5
| 2 16 3
4′ = ⅓ | 0 9
6″ = ⅛ | 0 1⅛
£2 17 1⅛

By fractions. 7 ft. 3 in. = 7¼ ft., and 3 ft. 6 in. = 3½ ft.

∴ Area = 7¼ × 3½ = $\frac{203}{8}$ = 25⅜ sq. ft.

∴ Cost = 2*s.* 3*d.* × 25⅜ = £2 17*s.* 1⅛*d.*

4. Required the area of a rectangular table 9 ft. 6 in. long, and 4 ft. 3 in. broad. *Ans.* 40 *ft.* 4′ 6″.

5. Required the cost, in the last example, at 1*s.* 6*d.* per sq. foot. *Ans.* £3 0*s.* 6$\frac{3}{4}$*d.*

6. What is the area of a square table, the side of which measures 3 ft. 8 in.? *Ans.* 13 *ft.* 5′ 4″.

7. A rectangular floor is 29 ft. 2 in. long, and 18 ft. 4 in. broad; what did it cost in paving at 6*s.* 9*d.* per sq. yd.? *Ans.* £20 1*s.* 0$\frac{1}{2}$*d.*

8. What is the area of a parallelogram, whose base is 6 ft. 4 in. and perpendicular height 3 ft. 5 in.? *Ans.* 21 *ft.* 7′ 8″.

9. Required the acreage of a field, in the form of a parallelogram, whose length or base is 930 links, and perpendicular breadth 480 links. *Ans.* 4 *ac.* 1 *r.* 34 *p.*

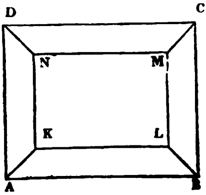
10. How many yards of carpet, 3 qrs. wide, will cover a room that measures 14 ft. 3 in. by 9 ft. 4$\frac{1}{2}$ in.? *Ans.* 19 *yds.* 2 *ft.* 4$\frac{1}{2}$ *in.*

11. How many square yards are there in a parallelogram whose length is 37 ft., and perpendicular breadth 2·625 ft.? *Ans.* 10·79.

12. How many acres are in a rectangular field, the length of which is 1375 links, and breadth 95 links? *Ans.* 1 *ac.* 1 *r.* 9 *p.*

13. Required the cost of a piece of ground 24 ft. 3 in. by 7 ft. 6 in., at 3*s.* 6*d.* per sq. foot. *Ans.* £31 16*s.* 6$\frac{3}{4}$*d.*

14. Let ALCN represent a picture frame, where the length AB is 20 in., the height AD 16 in., and the breadth of each side of the frame 3 in.; required the surface in the frame.



Here to find the surface in the frame, we shall subtract the area of the rectangle KLMN from ABCD.

Area rectangle ABCD $= 20 \times 16 = 320$ sq. in.

Now KL $= 20 - 6 = 14$, and KN $= 16 - 6 = 10$;

$\therefore$ Area rectangle KLMN $= 14 \times 10 = 140$ sq. in.

$\therefore$ Surface of the frame $= 320 - 140 = 180$ sq. in.

15. A coach road goes round the rectangular field ABCD; it is required to find the area of the ground taken up by the coach road, when its breadth is 4 yds.; the length AB of the field being 88 yds., and the breadth 40 yds.

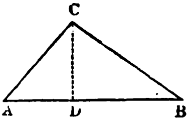
Ans. 960 *sq. yds.*

2. **Problem.** To find the area of a triangle when its base and perpendicular height are given.

Rule. Multiply the base by the perpendicular height, and half the product will be the area. See Geo. p. 35.

Examples.

1. Required the number of acres in the triangular field ABC, whose base AB is 945 links, and perpendicular CD 480 links.



$$\text{Area} = \frac{945 \times 480}{2} = 226800 \text{ sq. links}$$

$$= 2 \; ac. \; 1 \; r. \; 2 \; p.$$

2. Required the area of a triangular field, one of whose sides measures 618 links, and the perpendicular on it, from the opposite angle, 730 links. *Ans.* 2 *ac.* 1 *r.* ·912 *p.*

3. How many square yards of brick-work are in the gable top of a house, whose breadth is 20 ft. 6 in., and perpendicular height 10 ft. 4 in.? *Ans.* 11 *yds.* 6 *ft.* 11′.

4. How many square yards are there in a right-angled triangle, whose base is 49 ft., and perpendicular 25·25 ft.?

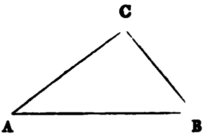
Ans. 68·736.

3. **Problem.** To find the area of a triangle, given the three sides.

Rule. From half the sum of the three sides subtract each side severally; multiply the half sum and the three remainders continually together, and the square root of the last product will be the area of the triangle. (See Trigo. Art. **36.**)

Note. The perpendicular let fall upon any side of a triangle may be found by dividing the area by half that side.

EXAMPLES.



1. In the triangular field ABC, the side AB measured 9 chains, AC 8·4, and BC 7·8 chains; how many acres does the field contain?

The half sum $=\frac{1}{2}(9+8\cdot4+7\cdot8)=12\cdot6$;

$$\begin{array}{cccl} 12\cdot6 & 12\cdot6 & 12\cdot6 & \\ 9 & 8\cdot4 & 7\cdot8 & \\ \hline 3\cdot6 & 4\cdot2 & 4\cdot8 & \text{the three remainders.} \end{array}$$

$\therefore$ Area $=\sqrt{12\cdot6 \times 3\cdot6 \times 4\cdot2 \times 4\cdot8}$ sq. ch. $=3$ *ac.* 3·84 *p.*

By logarithms.

$$\begin{array}{rl} \log 12\cdot6 = & 1\cdot10037 \\ ,, \quad 3\cdot6 = & 0\cdot55630 \\ ,, \quad 4\cdot2 = & 0\cdot62325 \\ ,, \quad 4\cdot8 = & 0\cdot68124 \\ \hline \end{array}$$

Dividing by the index of the root $2\overline{)2\cdot96116}$

log area triangle $=1\cdot48058=\log 30\cdot24$.

$\therefore$ Area triangle $=30\cdot24$ sq. ch. $=3$ *ac.* 3·84 *p.*

2. The sides of a triangle are 9, 8, and 7; required the area. *Ans.* 26·83.

3. The sides of a field are 620 links, 540 links, and 710 links; required the area. *Ans.* 1 *ac.* 2 *r.* 18 *p.*

4. The side of an equilateral triangle is 12 ft.; required its area. *Ans.* 62·353 *sq. ft.*

5. The sides of a triangular field measure 146·5, 119·5, and 92·5 yards; required its value at the rate of £370 per acre. *Ans.* £421 13*s.* 10*d.*

6. Required the area of a triangular field, whose sides are 469, 427·8, and 512·8 links. *Ans.* 3 *r.* 30·4 *p.*

7. In the triangle ABC (see fig. page 143.) given AB = 35, AC = 28, and BC = 21; required the perpendicular CD.

Here we first find the area to be 294; then we have,

$$\tfrac{1}{2} \times 35 \times \text{CD} = \text{area triangle} = 294;$$

$$\therefore\ \text{CD} = 294 \div \tfrac{1}{2} \times 35 = 16{\cdot}8.$$

8. Required the perpendicular let fall upon the least side in example 2. *Ans.* 7·66.

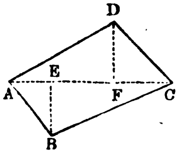
4. Problem. To find the area of a trapezium.

Rule. Multiply the diagonal by the sum of the perpendiculars let fall upon it, from the opposite angles, and half the product will be the area.

Or, divide the trapezium into triangles, then the sum of the areas of these triangles, calculated by Problem **2.** or **3.**, will be the area required. (See Geo. Art. **42.**)

Examples.

1. Required the area of the trapezium ABCD, whose diagonal AC is 65 ft., the perpendicular BE is 33·5 ft., and the perpendicular DF is 28 ft.



Here the sum of the perpendiculars

$$= 33{\cdot}5 + 28 = 61{\cdot}5;$$

$$\therefore\ \text{Area ABCD} = \frac{65 \times 61{\cdot}5}{2} = 1998{\cdot}75 \text{ sq. ft.}$$

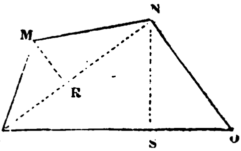
2. How many square yards of paving are in the trapezium ABCD, whose diagonal AC = 20 yds., and the perpendiculars BE = 4·2 yds., and DF = 3·8 yds.? *Ans.* 80 *sq. yds.*

3. In measuring the field ABCD (see the last fig.) I found AE = 125 links, EB = 280 links, AF = 540 links, FD = 420

links, and AC = 750 links. It is required to construct the figure, and find its area. *Ans.* 2 *ac.* 2 *r.* 20 *p.*

4. Required the area of the field ABCD, whose sides are AB = 600 links, BC = 480, CD = 320, and DA = 540, and the diagonal AC = 760 links. *Ans.* 2 *ac.* 0 *r.* 27 *p.*

5. In surveying the field PONM, I found PO = 1621·5, and the perpendicular NS = 584, PN = 1209, and the perpendicular MR = 305 links; required the area.
Ans. 6 *ac.* 2 *r.* 12·5 *p.*



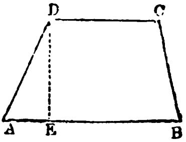
5. PROBLEM. To find the area of a trapezoid.

RULE. Multiply the sum of the two parallel sides by the perpendicular distance between them, and half the product will be the area. (See Geo. Art. **43.**)

Note. The area of any irregular figure may be found by dividing it into triangles or four-sided figures, or into both.

EXAMPLES.

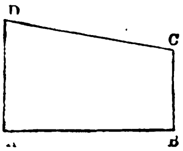
1. In the trapezoid ABCD, the parallel sides AB and DC are 4·6 and 3 chains, and the perpendicular distance DE = 6·037 chains; required the area.



The sum of the parallel sides
$= 4·6 + 3 = 7·6$;

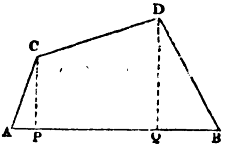
$$\therefore \text{ Area} = \frac{7·6 \times 6·037}{2} = 22·94 \text{ sq. ch.} = 2\,ac.\ 1\,r.\ 7\,p.$$

2. In the trapezoid ABCD, the parallel sides AD and BC are 80 and 60 links, and AB, the perpendicular distance between them, is 840 links; what is the area? *Ans.* 2 *r.* 14 *p.*



3. Given, as in the last example, AD = 80, BC = 20, and AB = 340 links, to find the area.
Ans. 27·2 *p.*

4. In measuring along one side AB of the field ABDC, I found AP = 110 links, the perpendicular CP = 352, AQ = 745, the perpendicular DQ = 595, and AB = 1110; required the area.



2 area triangle APC = 110 × 352 = 38720
„ trapezoid PQDC = 947 × 635 = 601345
„ triangle QBD = 595 × 365 = 217175

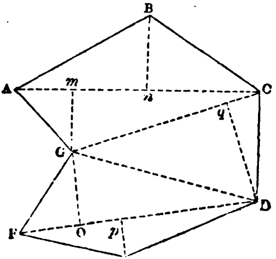
2857240

Area ABDC = 428620 sq. l.
= 4 *ac.* 1 *r.* 5·79 *p.*

5. Required the same as in the last example, when AP = 20, CP = 6, AQ = 30, DQ = 14, and AB = 38 feet.

Ans. 216 *sq. ft.*

6. Required the area of the irregular figure ABCD, &c., the following diagonals and perpendiculars being given, *viz.* AC = 27·5, B*n* = 9, G*m* = 6·5, GC = 22, D*q* = 11·5, FD = 26, G*o* = 6, and E*p* = 4.



2 area ABC = 27·5 × 9 = 247·5
„ AGC = 27·5 × 6·5 = 178·75
„ GDC = 22 × 11·5 = 253
„ FGD = 26 × 6 = 156
„ FDE = 26 × 4 = 104

2)939·25

Area ABCD, &c. = 469·625

7. Required the area of a pentangular figure ABCDGA; when the diagonal AC = 40, and the two perpendiculars, B*n*

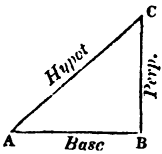
and Gm, upon it are 8 and 9; the diagonal GC = 38, and the perpendicular Dq upon it is 6. *Ans.* 454.

6. PROBLEM. Any two sides of a right-angled triangle being given, to find the third side.

RULE. To find the hypotenuse; add the squares of the two legs together, and extract the square root of the sum.

To find one leg; subtract the square of the given leg from the square of the hypotenuse, and extract the square root of the difference. (See Geo. Art. **45.**)

EXAMPLES.



1. In the right-angled triangle ABC, the base AB = 33 ft., and the perpendicular BC = 44 ft.; required the hypotenuse AC.

Here, the sum of the squares of the two legs $= 33^2 + 44^2 = 3025$;

$$\therefore \text{AC} = \sqrt{3025} = 55 \text{ ft.}$$

2. If AC = 65, and BC = 52; required AB.

$$\text{Here } 65^2 - 52^2 = 1521; \;\therefore\; \text{AB} = \sqrt{1521} = 39.$$

3. The legs of a right-angled triangle are 4·5 and 9; required the hypotenuse. *Ans.* 10·0623.

4. If the hypotenuse is 70, and the perpendicular 42; required the base. *Ans.* 56.

5. The side of a square is 5 ft.; what is its diagonal? *Ans.* 7·071 *ft.*

6. A rectangular table is 4 ft. 6 in. long, and 2 ft. broad; what is the length of its diagonal? *Ans.* 4·924 *ft.*

7. A line of 95 yds. will reach from the top of a castle, which stands at the side of a river, to the opposite bank; required the breadth of the river, the height of the castle being 32 yds. *Ans.* 89·44 *yds.*

8. A ladder 45 ft. long, being placed in a street, will exactly reach to a window 27 ft. from the ground on one side; and, upon being turned over without moving the foot, will reach a window 36 ft. high on the other side; required the breadth of the street. *Ans.* 63 *ft.*

9. The legs of a right-angled triangle are 50 and 37·5; required the perpendicular let fall upon the hypotenuse from the right angle. *Ans.* 30. (See Geo. Art. **46.**)

Note 1. By means of this problem we are enabled to find the area of a rectangle, when the diagonal and one of the sides are given.

10. The length AB of a rectangle ABCD is 8 ft., and the diagonal AC is 10 ft.; required the area. (See fig. p. 140.)

$$\text{Here, } BC = \sqrt{10^2 - 8^2} = 6;$$

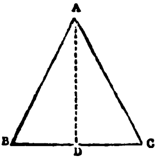
$$\therefore \text{ Area } ABCD = 8 \times 6 = 48 \text{ sq. ft.}$$

11. The diagonal of a square is 6 ft.; required the area. *Ans.* 18 *sq. ft.*

12. Find the area of a rectangle, whose base and diagonal are 36 and 45 ft. *Ans.* 972 *sq. ft.*

Note 2. The area of an isosceles, or equilateral triangle, may be readily found by means of this problem.

13. Required the area of the isosceles triangle ABC, when $BC = 24$ yds., and $AB = AC = 20$ yds.



Here the perpendicular AD bisects the base, therefore $BD = \frac{1}{2}$ of 24 yds. = 12 yds.; then from the right-angled triangle ABD we have,

$$AD = \sqrt{20^2 - 12^2} = 16;$$

$$\therefore \text{ Area triangle } ABC = BD \times AD = 12 \times 16 = 192 \text{ sq. yds.}$$

14. Required the area of an isosceles triangle, whose base is 48 ft., and each of the equal sides 40 ft. *Ans.* 768 *sq. ft.*

15. Required the area of an equilateral triangle, whose *side BC is 16.* (See last figure.) Ans. 110·85.

Here $BD = \frac{1}{2}$ of $16 = 8$, and $AD^2 = (2 \times 8)^2 - 8^2 = 8^2 \times 3$;

$$\therefore\ AD = 8\sqrt{3}, \text{ and}$$

$$\text{area } ABC = BD \times AD = 8 \times 8\sqrt{3} = 8^2\sqrt{3}.$$

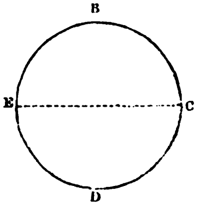
Here the 8 is half the given side, and the $\sqrt{3}$ will occur in every question which can be proposed; hence it follows that, *the area of an equilateral triangle is equal to the square of half the side multiplied by the square root of* 3.

16. Required the area of an equilateral triangle, whose side is 80 ft. *Ans.* 2771·28 *sq. ft.*

7. Problem. To find the diameter and circumference of a circle, the one from the other.

Rule. Multiply the diameter by 3·1416, and the product will be the circumference. And conversely, divide the circumference by 3·1416, and the quotient will be the diameter. (See Geo. Art. **74.**, and Trigo. Art. **34.**)

Examples.



1. If the diameter EC of a circle be 8, what is the length of the circumference EDCBE?

Circum. when the diam. is $1 = 3·1416$.

Circum. when the diam. is $8 = 8$ times $3·1416 = 25·1328$.

2. What is the circumference of the earth, supposing it to be perfectly round, and that its mean diameter is 7912 miles? *Ans.* 24856 *miles.*

3. The diameter of a well is 7 ft. 6 in.; what is its circumference? *Ans.* 23 *ft.* 6′ 8″.

4. The diameter of the planet Venus is 7680 miles; what is her circumference? *Ans.* 24127·488 *miles.*

5. The circumference EDCB is 35·5; required the diameter EC.

$$\text{Here, } EC \times 3·1416 = 35·5;$$

$\therefore$ EC $= 35{\cdot}5 \div 3{\cdot}1416 = 11{\cdot}29997.$

6. The circumference of the moon is 6850 miles; what is her diameter? *Ans.* 2180·4 *miles.*

7. The diameter of a circular plantation is 500 yds.; what did the wall, going round it, cost at 8*s.* 4*d.* per yard? *Ans.* £654 10*s.*

8. Problem. To find the length of any arc of a circle, when the diameter and number of degrees in the arc are given.

Rule. Find the circumference of the whole circle, multiply it by the number of degrees in the given arc, and divide this product by 360.

Note. When the chord and height are the given quantities, the radius and number of degrees must be found by Problem **15.**

Examples.

1. What is the length of an arc of 45° in a circle whose diameter is 4 ft.?

$$\text{Circum. whole circle, or } 360^\circ = 4 \times 3{\cdot}1416;$$

$$\therefore \quad " \quad \text{of } 1^\circ = \frac{4 \times 3{\cdot}1416}{360};$$

$$\therefore \quad " \quad \text{of } 45^\circ = \frac{4 \times 3{\cdot}1416 \times 45}{360} = 1{\cdot}5708\,ft.$$

2. Find the length of 30° in a circle whose diameter is 36 ft. *Ans.* 9·4248 *ft.*

3. What is the length of a circular arc of 32°, the radius of the circle being 5? *Ans.* 2·7925.

4. Find the length of an arc of 34° 20′, the diameter being 6.

$$\text{Here } 34^\circ\ 20' = 2060', \text{ and } 360^\circ = 21600'; \text{ then}$$

$$\text{Circum. whole circle, or } 21600' = 6 \times 3{\cdot}1416;$$

$$\therefore \quad " \quad \text{of} \quad 1' = \frac{6 \times 3{\cdot}1416}{21600};$$

$$\therefore \text{ Circum. of } 2060' = \frac{6 \times 3{\cdot}1416 \times 2060'}{21600} = 1{\cdot}797.$$

5. Find the length of an arc of 27° 18′, the diameter being 62·5 yds. *Ans.* 14·889 *yds.*

6. The diameter of a circle is 5 ft.; required the number of degrees in an arc whose length is 4 ft.

$$\text{Circum. circle, or arc of } 360^\circ = 5 \times 3{\cdot}1416\,;$$

$$\therefore \text{ arc of } 1^\circ = \frac{5 \times 3{\cdot}1416}{360}\,;$$

$$\therefore \text{ No. degrees in the given arc} = 4 \div \text{arc of } 1^\circ$$

$$= 4 \div \frac{5 \times 3{\cdot}1416}{360} = 91{\cdot}673^\circ = 91^\circ\ 40'\ 22''.$$

7. The diameter of a circle is 16; required the number of degrees in the arc whose length is 14. *Ans.* 100° 16′ 2″.

9. Problem. To find the area of a circle.

Rule. 1. Multiply half the circumference by half the diameter, and the product will be the area. (See Geo. Art. **75.**)

2. Multiply the square of the diameter by ·7854, and the product will be the area. (See Trigo. Art. **34.**)

Note. If the area be divided by ·7854, the quotient will be the square of the diameter.

Examples.

1. What is the area of a circle, whose circumference is 22 ft., and diameter 7 ft.?

$$\text{Here, area} = \tfrac{22}{2} \times \tfrac{7}{2} = 38{\cdot}5 \textit{ sq. ft.}$$

2. What is the area of a circle whose diameter is 7?

$$\text{Area} = 7^2 \times {\cdot}7854 = 38{\cdot}4846.$$

3. The diameter of a cylindrical vessel is 2 ft. 6 in.; what is the area of its bottom? *Ans.* 4·90875 *sq. ft.*

4. The diameter of a circular table is 4 ft. 3 in.; what is its area? *Ans.* 14·186 *sq. ft.*

5. The circumference of a stone column is 5 ft.; what is the area of its base?

$$\text{Here, diam. base} = \frac{5}{3\cdot1416};$$

$$\therefore \text{ Area} = \tfrac{1}{2} \text{ circum.} \times \text{rad.} = \frac{5}{2} \times \frac{1}{2} \times \frac{5}{3\cdot1416} = \frac{25}{4 \times 3\cdot1416}$$
$$= 1\cdot989 \textit{ sq. ft.}$$

6. What is the area of a circular plantation whose circumference is 960 links? *Ans.* 2 *r.* 37·3 *p.*

7. The area of a circle is 24 sq. ft.; required the diameter.

Here, $\text{diam.}^2 \times \cdot7854 = 24$;

$$\therefore \text{ diam.}^2 = 24 \div \cdot7854 = 30\cdot5576,$$

$$\therefore \text{ diam.} = \sqrt{30\cdot5576} = 5\cdot527 \textit{ ft.}$$

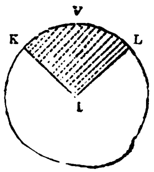
8. What length of cord will it take to tether an ass, so that he may graze over an acre of ground? *Ans.* 39·25 *yds.*

10. Problem. To find the area of a sector of a circle.

Rule. 1. Find the area of the whole circle, multiply it by the number of degrees in the arc, and this product divided by 360 will be the area of the sector. 2. Multiply half the length of the arc by the radius.

Note. When the radius and the arc, or the number of degrees in it, are not given, they must be found by Problem **15**.

Examples.



1. In the sector IKVL of a circle, the radius IK is 7 ft., and the number of degrees in the arc KVL is 29°; required the area of the sector.

The circle may evidently be cut into 360 sectors of one degree in the arc.

$$\text{Area whole circle or } 360^\circ = 14^2 \times \cdot 7854;$$

$$\therefore \text{ area sector of } 1^\circ = \frac{14^2 \times \cdot 7854}{360},$$

$$\therefore \quad ,, \quad ,, \quad 29^\circ = \frac{14^2 \times \cdot 7854 \times 29}{360} = 12 \cdot 4 \textit{ sq. ft.}$$

2. The arc of a sector contains 18°, and its diameter is 3 ft.; required its area. *Ans.* ·35343 *sq. ft.*

3. Required the area of a sector, whose arc contains 27° 40′, and its radius is 14. *Ans.* 47·3218.

4. The length of an arc of a sector is 8·6 inches, and the radius is 3 ft. 6 in.; what is its area? *Ans.* 1·254 *sq. ft.*

5. The radius of a sector is 25 ft., and the length of the arc 8 ft.; what is its area? *Ans.* 100 *sq. ft.*

6. The area of a sector is 9 sq. ft., and the diameter of the circle is 5 ft.; required the number of degrees in the arc of the sector.

$$\text{Area whole circle} = 5^2 \times \cdot 7854;$$

$$\therefore \text{ Area sector of } 1^\circ = \frac{5^2 \times \cdot 7854}{360};$$

$$\therefore \text{ No. degrees in the given sector} = 9 \div \text{area sector of } 1^\circ$$

$$= 9 \div \frac{5^2 \times \cdot 7854}{360} = 165^\circ\ 0'\ 4''.$$

7. The area of a sector is 18 sq. ft., and the diameter of the circle is 9 ft.; required the number of degrees in the arc of the sector. *Ans.* 101° 51′ 32″.

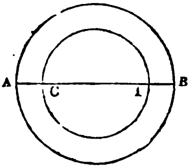
8. The area of a sector is 28·4 sq. in., and the radius of the circle is 8 in.; required the length of the arc of the sector. *Ans.* 7·1 *in.*

11. Problem. To find the area of a circular ring, or space included between the circumferences of two concentric circles.

Rule. The difference of the areas of the two circles will *be the* area of the ring. Or, multiply the sum of the diameters by their difference, and this product multiplied by ·7854 will give the required area.

EXAMPLES.

1. The diameters AB and CD are 5 and 3; required the area of the circular ring.



$$\text{Area circle AB} = 5^2 \times {\cdot}7854;$$

$$\text{,,} \quad \text{,,} \quad \text{CD} = 3^2 \times {\cdot}7854;$$

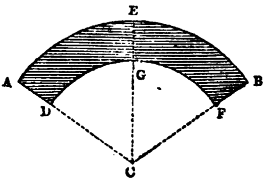
$$\therefore \text{ Area ring} = 5^2 \times {\cdot}7854 - 3^2 \times {\cdot}7854 = (5^2 - 3^2) \times {\cdot}7854$$
$$= (5 + 3)\,(5 - 3) \times {\cdot}7854 = 12{\cdot}5664.$$

2. The diameters of the circles are 10 and 6; required the area of the ring. *Ans.* 50·2656.

3. The exterior diameter of a metal pipe is 3 in., the interior diameter 2 in.; what is the area of the circular ring in the section? *Ans.* 3·927 *sq. in.*

4. The interior diameter of a circular building is 48 ft., and the thickness of the wall 1 ft.; required the area of the ground occupied by the wall. *Ans.* 153·9384 *sq. ft.*

5. Required the number of square feet in the face of the circular arch ABFD, when the radius AC = 10 ft., DC = 9 ft., and the length of the arc AEB = 20 ft.



To find the length of the arc DGF, we have,

$$\text{Length arc to rad. } 10 \text{ ft.} = 20 \text{ ft.};$$

$$\therefore \quad \text{,,} \quad \text{,,} \quad \text{,,} \quad 1 \text{ ft.} = \tfrac{20}{10} = 2 \text{ ft.};$$

$$\therefore \quad \text{,,} \quad \text{,,} \quad \text{,,} \quad 9 \text{ ft.} = 9 \text{ times } 2 \text{ ft.} = 18 \text{ ft.}$$

$$\begin{array}{ll} \text{Hence, area sector } ACB = \frac{1}{2} \times 20 \times 10 = 100, \\ \quad\;\; \text{"} \quad\;\; \text{"} \quad DCF = \frac{1}{2} \times 18 \times 9 = 81, \\ \therefore \text{ Area sectoral ring } ADFB = 100 - 81 = 19 \textit{ sq. ft.} \end{array}$$

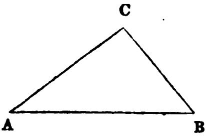
6. Required the same as in the last example, when $AC = 12$, $DC = 10$, and the arc $AEB = 36$ ft. *Ans.* 66 *sq. ft.*

12. PROBLEM. To find the area of a triangle, or a parallelogram, given two sides and the included angle.

RULE. The area of a triangle is equal to half the product of the two given sides and the natural sine of the included angle. The area of a parallelogram is equal to the product of the two given sides and the natural sine of the included angle. (See Trigo. Art. **12.**)

EXAMPLES.

1. What is the area of the triangle ABC, when $AB = 72$, $AC = 25{\cdot}5$, and the included angle $A = 58°$?



Here, area $= \frac{1}{2} \times 72 \times 25{\cdot}5 \times \sin 58° = 918 \times {\cdot}8480481 = 778{\cdot}5$.

By logarithms.

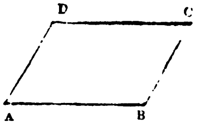
$$\begin{array}{lcl} \log. 36 & = & 1{\cdot}5563025 \\ \text{"}\; 25{\cdot}5 & = & 1{\cdot}4065402 \\ \text{"}\; \sin 58° & = & 9{\cdot}9284205 \\ & & -10{\cdot} \\ \hline \text{"}\; \text{area} & = & 2{\cdot}8912632 = \log 778{\cdot}5. \end{array}$$

2. What is the area of a triangle, two of whose sides are 45 and 40 ft., and the included angle 28°?

Ans. 422·524 *sq. ft.*

3. How many acres are contained in a triangular field, two of whose sides are 300 and 400 links, and the included *angle* $28° 57'$? *Ans.* 1 *r.* 6·47 *p.*

4. What is the area of a field in the form of a parallelogram, ABCD, the two adjacent sides AB and AD being 724 and 586 links, and the included angle A = 57°? *Ans.* 3 *ac.* 2 *r.* 9 *p.*



5. Required the same as in the last example, when AB = 25·35, AD = 2·08 chains, and ∠ A = 30°.

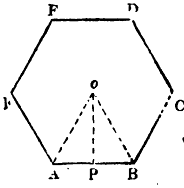
Ans. 2 *ac.* 2 *r.* 21·8 *p.*

13. Problem. To find the area of any regular polygon.

Rule. Multiply the sum of the sides of the polygon by the perpendicular let fall from its centre upon one of the sides, and half the product will be the area. (See Geo. Art. **71.**) When the side only is given, the area may be found by the formula of Art. **39.** p. 130.

Examples.

1. Required the area of the regular hexagon ABCDEF, whose side AB is 7·3 ft., and the perpendicular OP is 6·32 ft.



Here, the sum of the sides = 6 × 7·3,

$$\therefore \text{ Area} = \frac{6 \times 7{\cdot}3 \times 6{\cdot}32}{2} = 138{\cdot}408\, sq.ft.$$

2. Required the area of a regular octagon, whose side is 4·9705, and the perpendicular 6. *Ans.* 119·292.

3. Required the area of a heptagonal garden, whose side measures 17 ft. 6 in., and perpendicular 18 ft.

Ans. 1102·5 *sq. ft.*

4. Required the area of a regular hexagon, whose side is 3.

Here in the formula, Trigo. Art. **39.**, the side = 3, the number of sides $n = 6$, and $\therefore \frac{180^\circ}{n} = 30^\circ$, then,

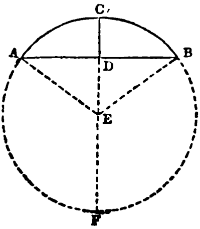
$$\text{Area polygon} = \tfrac{6}{4} \times 3^2 \times \text{cot. } 30^\circ = 23{\cdot}38.$$

Or thus, without the formula. In this case the triangle AOB′ (see last fig.) is equilateral; therefore (Note 2. Problem **6.**) area $AOB = 1{\cdot}5^2 \times \sqrt{3}$, and area polygon

$$= 6 \times 1{\cdot}5^2 \times \sqrt{3} = 23{\cdot}38.$$

5. Required the area of a regular octagon, whose side is 8 ft. *Ans.* 309·019 *sq. ft.*

14. PROBLEM. To find the area of a segment of a circle.



RULE. Find the area of the sector ACBE; also find the area of the triangle ABE; then the difference of these areas, will be the area of the segment ABC, when it is less than the semicircle. The sector must be added to the triangle when the segment is greater than the semicircle.

Note. When the radius and the number of degrees in the arc are not given, they must be found by Problem **15.**

EXAMPLES.

1. What is the area of the segment ABC, whose radius AE = 24, and the arc ACB = 37°?

Here we find by Problem **10.**,

$$\text{Area sector ACBE} = \frac{48^2 \times {\cdot}7854 \times 37}{360} = 185{\cdot}9;$$

and by Problem **12.**, we find,

$$\text{Area triangle ABE} = \tfrac{1}{2} \times 24 \times 24 \times \sin 37^\circ = 173{\cdot}3;$$

$$\therefore \text{ Area segment ABC} = \text{sector} - \text{triangle}$$
$$= 185{\cdot}9 - 173{\cdot}3 = 12{\cdot}6.$$

2. What is the area of a segment, whose arc is 60°, and the radius of the circle 5 ft.? *Ans.* 2·2647 *sq. ft.*

3. What is the area of a segment, whose arc is 90°, and radius 9 ft.? *Ans.* 23·117 *ft.*

4. Required the area of a segment, whose arc contains 280°, the diameter being 25 yds. *Ans.* 458·729 *sq. yds.*

15. PROBLEM. The chord and height of an arc of a circle being given, to find the radius and number of degrees in the arc. (See the last figure.)

By Geo. Cor. 1. Art. **61.**, $\text{CD} \cdot \text{DF} = \text{AD}^2$, $\therefore \text{DF} = \frac{\text{AD}^2}{\text{CD}}$; adding CD to each side of this equality,

$$\text{CF or diameter} = \frac{\text{AD}^2}{\text{CD}} + \text{CD}.$$

Having found the diameter, we now proceed to find the number of degrees in the arc.

By Trigo. Art. **9.**, $\sin \text{AEC}$ or $\sin \frac{1}{2}\,\text{AEB} = \frac{\text{AD}}{\text{AE}}$; hence from the table of sines, we find $\angle\,\text{AEC}$, and $\therefore \angle\,\text{AEB}$.

EXAMPLES.

1. Given the chord $\text{AB} = 8$, and the height $\text{CD} = 2$, to find the radius of the circle, and the number of degrees in the arc ACB.

Here, $\text{AD} = \frac{1}{2}$ of $8 = 4$; and $2 \times \text{DF} = 4^2$; $\therefore \text{DF} = 8$, and CF or diameter $= 8 + 2 = 10$, and the radius $= 5$.

From the right-angled triangle ADE, we have,

$$\sin \text{AEC} = \frac{\text{AD}}{\text{AE}} = \frac{4}{5} = \cdot 8, \ \therefore \angle\,\text{AEC} = 53^\circ\ 8' \text{ nearly, and}$$

$\angle\,\text{AEB}$, or degrees in the arc $\text{ACB} = 53^\circ\ 8' \times 2 = 106^\circ\ 16'$.

2. Required the same as in the last example, when $\text{AB} = 12$, and $\text{CD} = 4$.

Ans. radius $= 6{\cdot}5$, *and arc* $= 134^\circ\ 45'$.

3. Given the height $\text{CD} = 6$, and the chord of the half arc

AC $= 10$, to find the radius and number of degrees in the arc. *Ans. radius* $= 8\frac{1}{3}$, *and arc* $= 147° 28′$

Here, $AD^2 = 10^2 - 6^2 = 64$; then $DF = \frac{64}{6} = 10\frac{2}{3}$; and so on, as in the first example.

4. Required the same as in the last example, when $CD = 9$, and the chord of half the arc $= 12$.

Ans. radius $= 8$, *and arc* $= 194° 22′$.

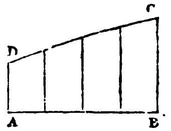
16. PROBLEM. To find the area of any space by means of equidistant ordinates or perpendiculars.

RULE. To the two extreme ordinates add twice the sum of the intermediate ordinates, and this sum multiplied by half the common distance between them will be the area.

Note. When the curvilineal boundary in the figure is some continuous curve, the area may be more accurately found by Thomas Simpson's rule.

EXAMPLES.

1. Required the area of the curvilineal space ABCD, where the extreme ordinates or perpendiculars AD and BC are 4 and 12 ft. respectively, the three intermediate ordinates, taken in order, are 6, 8, and 10 ft., and the common distance between them is 4 ft.



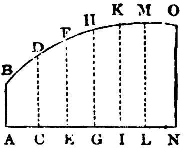
Here, taking the curve lying between any two consecutive ordinates as a straight line, we may consider the figure as being made up of four trapezoids, whose areas are as follows: area 1st $= \frac{4}{2}(4+6)$; area 2d $= \frac{4}{2}(6+8)$; area 3d $= \frac{4}{2}(8+10)$; and area 4th $= \frac{4}{2}(10+12)$;

$$\therefore \text{ Total area} = \tfrac{4}{2}(4+6) + \tfrac{4}{2}(6+8) + \tfrac{4}{2}(8+10) + \tfrac{4}{2}(10+12)$$
$$= \tfrac{4}{2}\{4 + 12 + 2(6 + 8 + 10)\} = 128 \textit{ sq. ft.}$$

2. Required the same as in the last example, when

AD = 8 yds., BC = 21 yds., the three intermediate ordinates, taken in order, 11, 14, and 17 yds., and the common distance between them 6 yds. *Ans.* 339 *sq. yds.*

3. Required the area of the curvilineal space ABON where the lengths of the equidistant ordinates are as follows: AB = 2, CD = 3, EF = 5, GH = 6, IK = 9, LM = 10, and NO = 10·5 ft.; and their distance apart, AC = CE = &c. = ·5 ft.

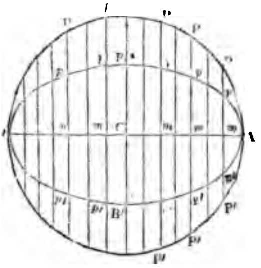


Ans. 19·625 *sq. ft.*

17. Problem. To find the area of an ellipse.

Rule. Multiply the product of the two diameters by ·7854, and the result will be the area.

Let ABA′B′ be an ellipse whose major diameter is AA′ and minor diameter BB′. Let APA′P′ be a circle described upon AA′ as a diameter, and let P*m* and *pm* be any corresponding ordinates of the circle and the ellipse; then it is one of the most remarkable properties of these curves, that the ratio of P*m* to *pm* is the same as the ratio of AA′ to BB′. Now if the ellipse and circle be divided into a series of very narrow bands, as shown in the figure, the area of *any* one of the circular bands will be to the area of the corresponding elliptical band in the ratio of P*m* to *pm* or AA′ to BB′; hence it follows that the area of the whole circle will be to the area of the whole ellipse in this same ratio, that is,

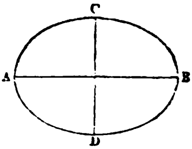


$$\text{Area circle} : \text{area ellipse} :: AA' : BB',$$

$$\begin{aligned} \therefore \text{ Area ellipse} &= \text{area circle} \times BB' \div AA' \\ &= \cdot7854 \times AA'^2 \times BB' \div AA' \\ &= AA' \times BB' \times \cdot7854; \end{aligned}$$

which is the analytical expression of the rule.

EXAMPLES.



1. What is the area of the ellipse ACBD, whose major diameter AB is 16 ft., and minor diameter CD 12 ft.?

Here, area ACBD $= 16 \times 12 \times \cdot 7854$
$= 150\cdot 7968$ *sq. ft.*

2. The diameters of an ellipse are 18 and 8 ft.; required the area. *Ans.* 113·0976 *sq. ft.*

3. The diameters of an elliptical piece of ground are 3 and 2 chains; required the area. *Ans.* 1 *r.* 35·3984 *p.*

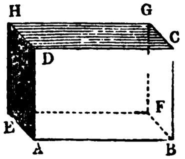
MENSURATION OF SOLIDS.

18. PROBLEM. To find the solidity of a rectangular solid or right-angled parallelopiped.

RULE. Multiply the length by the breadth, and that product again by the depth or altitude. (See Geo. Art. **90.**)

Note. To find the surface of a rectangular solid, multiply the perimeter of its end by its length, and the product will be the area of the sides; to which add twice the area of its end, and the sum will be the whole surface.

EXAMPLES.



1. Find the number of cubic feet in a rectangular solid ABGE, whose length AB is 6 ft. 3 in., breadth AE 4 ft. 6 in., and depth or altitude AD 3 ft. 1 in.

By decimals.		*By duodecimals.*
6·25 ft.		6 ft. 3′
4·5		4 6
3125		25 ·0
2500		3 1 6
28·125 area base		28 1 6
$3\frac{1}{12}$		3 1
84·375		84 4 6
2·3437		2 4 1 6
86·7187 *c. ft.*	*Ans.*	86 *c. ft.* 8′ 7″ 6‴

Here the 8′, instead of being cubic inches, denote 8 twelfths of a cubic foot; and the 7″ denote 144ths, and so on.

By fractions. Solid content $= 6\frac{1}{4} \times 4\frac{1}{2} \times 3\frac{1}{12} = \frac{25}{4} \times \frac{9}{2} \times \frac{37}{12}$ $= 86\frac{23}{32}$ *c. ft.*

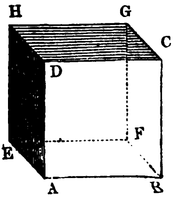
2. What is the solidity of a parallelopiped, whose length is 20 ft., breadth 5·75 ft., and depth 3·5 ft.? *Ans.* 402·5 *c. ft.*

3. Required the cost of a block of stone 8 ft. 4 in. long, 2 ft. 6 in. broad, and 1 ft. 3 in. deep; at 10*d.* per cubic foot. *Ans.* £1 1*s.* 8¼*d.*

4. What will be the cost of a log of timber 18 ft. long, 1 ft. 8 in. broad, and 1 ft. 6 in. deep; at 2*s.* 4*d.* per cubic foot? *Ans.* £5 5*s.*

5. What will be the cost of cutting a drain 60 ft. long, 5 ft. 6 in. broad, and 10 ft. 4 in. deep; at 8*d.* per cubic yard? *Ans.* £4 4*s.* 2¼*d.*

6. In the cube ABCE, the side AB or AD or AE is 1·75 ft.; required its solidity.



Here, as the length, breadth, and depth are the same, we have,

$$\text{Solidity} = 1{\cdot}75 \times 1{\cdot}75 \times 1{\cdot}75,$$
$$\text{or, } 1{\cdot}75^3 = 5{\cdot}359 \textit{ c. ft.}$$

By logarithms.

$$
\begin{array}{rr}
\log 1{\cdot}75 = & 0{\cdot}24304 \\
\text{Multiplying by the power} & 3 \\
\hline
\log \text{solidity} = & \underline{0{\cdot}72912} = \log 5{\cdot}359.
\end{array}
$$

7. The side of a cube is 2 ft. 6 in.; what is its solidity? *Ans.* 15·625 *c. ft.*

8. A cistern is 7 ft. 6 in. long, 1 ft. 9 in. broad, and 3 ft. 6 in. deep; how many cubic feet of water will it contain? *Ans.* 45·937.

9. What will be the cost of hewing all the faces of the stone in example 3; at 5*d.* per square foot? *Ans.* £1 8*s.* $7\frac{3}{4}$*d.*

10. A box, open at the top, is 4 ft. long, 2 ft. broad, and 3 ft. deep, and the thickness of the deal is $1\frac{1}{2}$ inches; required the number of cubic feet of timber in it.

In this case it will be most convenient to find the content of the exterior solid, and then subtract from this the content of the interior part of the box.

Content of exterior solid $= 4 \times 2 \times 3 = 24$ c. ft.

Now the interior length $= 4 - \frac{1}{4} = 3\frac{3}{4}$, breadth $= 2 - \frac{1}{4} = 1\frac{3}{4}$, and depth $= 3 - \frac{1}{8} = 2\frac{7}{8}$ ft.

$\therefore$ Content of the inside $= 3\frac{3}{4} \times 1\frac{3}{4} \times 2\frac{7}{8} = 18{\cdot}0859$ *c. ft.*

$\therefore$ Content of the timber $= 24 - 18{\cdot}0859 = 5{\cdot}914$ *c. ft.*

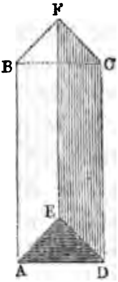
11. Required the same as in the last example, when the length is 3 ft., breadth $2\frac{1}{2}$ ft., depth 2 ft., and the thickness of the deal 3 inches. *Ans.* $6\frac{1}{4}$ *c. ft.*

19. Problem. To find the solidity of a prism, or of a cylinder.

Rule. Multiply the area of the base by the perpendicular height, and the product will be the solidity. (See Geo. Art. **90.**)

Note. To find the convex surface of a cylinder, multiply the *circumference* of the base by the height.

EXAMPLES.



1. What is the solidity of the triangular prism ADFB, whose length AB is 20 ft., and each of the equal sides of the base ADE 2 ft. 6 in.?

By Problem 3., area base

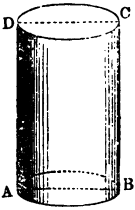
$= \sqrt{3 \cdot 75 \times 1 \cdot 25 \times 1 \cdot 25 \times 1 \cdot 25} = 2 \cdot 7063$;

$\therefore$ Solidity $= 2 \cdot 7063 \times 20 = 54 \cdot 126$ *c. ft.*

2. Required the same as in the last example, when AB = 14 ft. 3 in., and AD = AE = DE = 2 ft. *Ans.* 24·68 *c. ft.*

3. Find the solidity of a square prism, whose length is 41 ft., and one side of the base 1·25 ft. *Ans.* 64·062 *c. ft.*

4. What is the solidity of a prism whose base is a hexagon, each of the equal sides being $1\frac{1}{3}$ ft., and the length of the prism $7\frac{1}{2}$ ft.? *Ans.* 34·641 *c. ft.*



5. What is the solidity of the cylinder ABCD, when the diameter of the base AB is 30 in., and the height BC 25 in.?

Area base $= 30^2 \times \cdot 7854 = 706 \cdot 86$;

$\therefore$ Solidity $= 706 \cdot 86 \times 25 = 17671 \cdot 5$ *c. in.*

6. What is the solidity of a cylinder whose height is 2 ft. 6 in., and the diameter 4 ft.? *Ans.* 31·41 *c. ft.*

7. The diameter of a well is 3·75 ft., and its depth 22·5 ft.; what did it cost sinking at 3*s.* $7\frac{1}{2}$*d.* per cubic yard?

Ans. £1 13*s.* $4\frac{1}{4}$*d.*

8. The exterior diameter of a metal pipe is 6 in., the thickness of the metal $\frac{1}{2}$ in., and the length of the pipe is 50 ft.; required the number of cubic feet of metal in the pipe.

By Art. **11.**, area ring $=(6+5)(6-5)\cdot 7854=8\cdot 6394$ c. in.

$$\therefore \text{ Solidity} = \frac{8\cdot 6394}{144} \times 50 = 3 \textit{ c. ft. nearly.}$$

9. The length of a hollow iron roller is 4 ft., the exterior diameter 2 ft., and the thickness of the metal $\frac{3}{4}$ in.; what is its solidity? *Ans.* 1·5217 *c. ft.*

10. Required the convex surface of the cylinder in example 5.

It is evident that a piece of paper wrapped round the cylinder will have the shape of a rectangle, whose length is the height of the cylinder, and breadth the circumference of the base; hence we have,

$$\text{Circum. base} = 30 \times 3\cdot 1416 = 94\cdot 248\,;$$

$$\therefore \text{ Convex surface} = 94\cdot 248 \times 25 = 2356\cdot 2 \textit{ sq. in.}$$

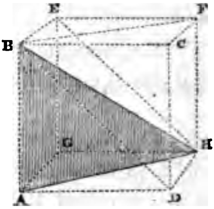
11. Required the convex surface of the cylinder in example 9., and also the number of revolutions it must make in rolling over an acre of ground.

Ans. convex surface=25·1328 *sq. ft., and no. revo.*=1733·19.

12. What is the convex surface of an upright cylinder, the diameter of whose base is 15 in., and the length 5 ft.? *Ans.* 2827·44 *sq. in.*

13. The circumference of the base of a round pillar is 10 ft., and the perpendicular height 38·636 ft.; what is its solidity? *Ans.* 307·46 *c. ft.*

14. The diagonal BH of the cube ADFB is 3; required its solidity.



Here the triangles ADH and HAB will be right-angled triangles,

$$\therefore \text{AH}^2 = 2\,\text{AD}^2,$$

$$\text{and } \text{BH}^2 = \text{AB}^2 + \text{AH}^2$$

$$= \text{AB}^2 + 2\,\text{AD}^2 = 3\,\text{AD}^2\,;$$

$$\therefore 3\,\text{AD}^2 = 3^2, \text{ and } \text{AD} = \sqrt{3},$$

$$\therefore \text{AD}^3, \text{ or content cube} = 3\sqrt{3}.$$

15. The length AD of a rectangular solid is 9, the height AB 3, and the diagonal BH 10; required the solidity.

Ans. 85·38.

20. Problem. To find the solidity of a cone, or pyramid.

Rule. Multiply the area of the base by the perpendicular height, and one-third the product will be the solidity. (See Geo. Art. **92.**)

Note. To find the surface, multiply the girt of the base by half the slant height; to this product add the area of the base, and it will give the whole surface.

Examples.

1. Required the solidity of the cone ABCE, when the diameter AC of the base is 4 ft., and the perpendicular height DE is 9 ft.

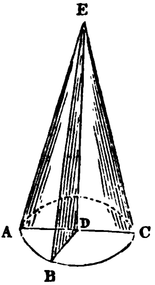
Here, area base $= 4^2 \times {\cdot}7854$;

$\therefore$ Solidity $=$ area base $\times$ perpend. $\times \frac{1}{3}$

$= 4^2 \times {\cdot}7854 \times \frac{9}{3} = 37{\cdot}699$ *c. ft.*

2. Required the same as in the last example, when AC $= 10$, and DE $= 24$.

Ans. 628·32.



3. Required the convex surface of the right cone in example 1.

If the cone ABCE be rolled on a plane, it will describe a circle about its vertex E as a centre, and with the slant height AE as a radius; hence the convex surface of the cone must form a *sector* of that circle, having the length of its arc equal to the circumference of the base of the cone. Or if the sector KIL be cut out of paper (see fig. p. 158.) and folded so as to bring the edges IL and IK together, the figure thus formed will be a cone; hence &c.

$\therefore$ Circum. base of the cone $= 4 \times 3{\cdot}1416 = 12{\cdot}5664$,

Slant height AE $= \sqrt{\text{AD}^2 + \text{DE}^2} = \sqrt{2^2 + 9^2} = 9{\cdot}2195$;

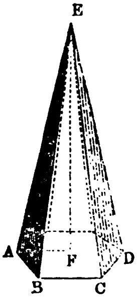
$\therefore$ Convex surface of the cone $= 12{\cdot}5664 \times 9{\cdot}2195 \times \frac{1}{2}$

$= 57{\cdot}929$.

4. The diameter of the base of an upright cone is 2·25 ft., and the slant height 20 ft.; required the convex surface.

Ans. 70·686 *sq. ft.*

5. Required the solidity of the hexagonal pyramid ABDE, each of the equal sides of the base being 40, and the perpendicular height EF 30.



Here we find by Problem **13.**,

$$\text{Area base} = 4156{\cdot}9216\,;$$

$$\therefore\ \text{Solidity} = 4156{\cdot}9216 \times 30 \times \tfrac{1}{3}$$

$$= 41569{\cdot}216.$$

6. What is the solidity of a triangular pyramid, whose perpendicular height is 24 ft., and the sides of the base 1·5, 2, and 2·5 ft.?

Ans. 12 *c. ft.*

7. Required the solidity of a pyramid, with a square base, the side of which is 9 ft., and the perpendicular height of the pyramid 15·3 ft. *Ans.* 413·1 *c. ft.*

8. In the right pyramid ABDE (see fig. to Ex. 5.), each of the equal sides of the base is 8, and the length of the edge AE = BE = &c. = 10; required the surface of the slant faces, and also the solidity of the pyramid.

Here, by Problem **3.**, we find,

$$\text{Area triangle ABE} = \sqrt{14 \times 6 \times 4 \times 4} = 36{\cdot}66,$$

but as there are 6 of these triangles,

$$\therefore\ \text{Area slant faces} = 36{\cdot}66 \times 6 = 219{\cdot}96.$$

Again, because ABCD is a hexagon, AF = AB = 8; therefore from the right-angled triangle AFE, we have,

$$\text{FE} = \sqrt{10^2 - 8^2} = 6\,;$$

$$\therefore\ \text{Solidity} = \text{area base} \times 6 \times \tfrac{1}{3} = 166{\cdot}277 \times 2 = 332{\cdot}55.$$

9. Required the same as in the last example, when AB = 2 ft., and AE = 2·5 ft.

Ans. 13·74 *sq. ft., and* 5·196 *c. ft.*

10. If the circumference of the base of a right cone is 40 ft., and height 25 ft., required its solidity, and also its whole surface. *Ans.* 1061 *c. ft. and* 643·2 *sq. ft.*

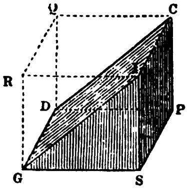
21. Problem. To find the solidity of a wedge.

Rule. To twice the length of the base, add the length of the edge, multiply this sum by the product of the breadth of the base and the height of the wedge, and $\frac{1}{6}$ of the last product will be the solidity.

Note. When the length of the edge is equal to the length of the base, the solidity will evidently be equal to half a prism of the same base and height.

Examples.

1. Required the solidity of the rectangular wedge GSCD, where the sides of the base $GS = 10$, and SP or $IC = 7$, and the perpendicular height $SI = 8$.

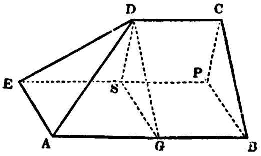


Here the wedge will be a prism and equal to half the rectangular solid GSPQR; hence we have,

$$\text{Solidity GSPQR} = 10 \times 7 \times 8 = 560;$$

$$\therefore \text{ Solidity wedge} = \tfrac{1}{2} \text{ of } 560 = 280.$$

2. Required the solidity of the wedge ABCDE, when the length $AB = 8$, the breadth $AE = 6$, the length of the edge $DC = 5$, and the perpendicular height of the wedge $= 9$.



Let DGS be a plane section parallel to the end CBP; then DPGS will be a prism, and ASDG will be a pyramid. Now $GB = DC = 5$, and $\therefore AG = 8 - 5 = 3$; hence we have,

$$\text{Solidity prism DPGS} = \tfrac{1}{2} \times 5 \times 6 \times 9 = 135,$$

$$\text{Solidity pyramid ASDG} = \tfrac{1}{3} \times 3 \times 6 \times 9 = 54;$$

$$\therefore \text{ Solidity whole wedge} = 135 + 54 = 189.$$

Or thus, by the rule: —

$$\text{Solidity wedge} = \tfrac{1}{6} \times 6 \times 9 \times (5 + 2 \times 8) = 189.$$

Let us now solve the question generally; for this purpose, put L = the length AB, l = the length of the edge DC, B = the breadth of the base, and h = the height of the wedge; and therefore AG = AB − GB = L − l; hence we have,

$$\text{Solidity prism DPGS} = \tfrac{1}{2} l \text{B} h,$$

$$\text{Solidity pyramid ASDG} = \tfrac{1}{3} \times \text{AG} \times \text{B} \times h = \tfrac{1}{3}(\text{L} - l)\,\text{B}h;$$

$$\therefore \text{ Solidity wedge} = \tfrac{1}{2} l\text{B}h + \tfrac{1}{3}(\text{L} - l)\,\text{B}h = \tfrac{1}{6}\text{B}h\,(l + 2\,\text{L});$$

which is the symbolical expression of the general rule.

If the edge were longer than the base, the formula would be the same.

3. The length and breadth of the base of a wedge are 64 and 9 in., the length of the edge 42 in., and the height 56 in.; what is the solidity? *Ans.* 14280 *c. in.*

4. The length and breadth of the base of a wedge are 35 and 15 in., the length of the edge 55 in., and the height 8·5725 in.; what is the solidity? *Ans.* 1·5503 *c. ft.*

22. Problem. To find the solidity of a rectangular prismoid, and also a frustum of a cone or pyramid.

Definition. A rectangular prismoid has its two parallel faces, ABPE and DCHF, rectangles. A frustum is a cone or pyramid having its top cut off.

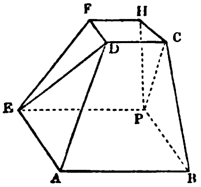
General Rule. To the sum of the areas of the two ends, add four times the area of the middle or mean section parallel to the ends, multiply this sum by the height, and one-sixth the product will be the solidity.

Note. To find the convex surface of a frustum of a cone, multiply half the sum of the perimeters of the two ends by the slant height. The faces of a prismoid have, in general, the form of a trapezoid.

EXAMPLES.

1. Required the solidity of the rectangular prismoid *ABCHE*, when the length of the base AB or EP = 12, and

the breadth AE or BP $=7$, the length of the top DC or FH $=8$, and the breadth DF or CH $=4$, and the height $=6$.



Here the prismoid will be divided into two wedges by a section EDCP passing through the opposite edges EP and DC; therefore by the preceding problem,

$$\text{Solidity wedge ABCE} = \tfrac{1}{6} \times 7 \times 6(8 + 2 \times 12) = 224,$$

$$\text{,,} \quad \text{,,} \quad \text{FHCE} = \tfrac{1}{6} \times 4 \times 6(12 + 2 \times 8) = 112;$$

$$\therefore \text{ Solidity prismoid} = 224 + 112 = 336.$$

Or by the rule. Here the length of the middle or mean section $= \frac{1}{2}(12+8) = 10$, and the breadth of it $= \frac{1}{2}(7+4) = 5{\cdot}5$;

$\therefore$ Solidity prismoid

$$= (12 \times 7 + 8 \times 4 + 4 \times 10 \times 5{\cdot}5) \times 6 \times \tfrac{1}{6} = 336.$$

Let us now prove the general rule; for this purpose, put $L =$ AB or EP, $l =$ DC or FH, $B =$ AE or BP, $b =$ DF or CH, and $h =$ the height of the prismoid. Then,

$$\therefore \text{ Solidity wedge ABCE} = \tfrac{1}{6} Bh\,(l + 2L),$$

$$\text{,,} \quad \text{,,} \quad \text{FHCE} = \tfrac{1}{6} bh\,(L + 2l).$$

Therefore by adding these expressions and reducing, we have,

$$\text{Solidity prismoid} = \tfrac{1}{6} h\,\{LB + lb + (L + l)\,(B + b)\};$$

putting $M = \frac{1}{2}(L + l)$ the mean or middle length, and $m = \frac{1}{2}(B + b)$ the mean or middle breadth, we finally have,

$$\text{Solidity prismoid} = \tfrac{1}{6} h\,(LB + lb + 4Mm);$$

which is the analytical expression of the general rule.

A section passing through ADHP, will evidently divide the prismoid into two equal prismoidal solids, having triangular bases; the rule, therefore, will hold true for all triangular prismoids. Now as every prismoid, as well as all frustums, may be regarded as being made up of triangular prismoids, it follows that the foregoing prismoidal formula will apply to all such solids.

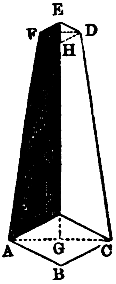
2. How many solid feet of timber are in a tree, whose ends are rectangles, the length and breadth of one end are 15 and 12 in., and the corresponding sides of the other end are 8 and 6 in., and the perpendicular length is 24 ft.?

Ans. 17·833 *c. ft.*

3. What is the solidity of the frustum ABCDF of a square pyramid; the side AB of the base being 3 ft., the side DE of the top 2 ft., and the perpendicular height GH 10 ft.?

Here the areas of the ends are 9 and 4 sq. ft., the side of the middle section $=\frac{1}{2}(3+2)=\frac{5}{2}$, therefore its area $=\frac{25}{4}$ sq. ft.; hence by the rule,

Solidity $=\frac{1}{6}\times 10\,(9+4+4\times\frac{25}{4})=63\frac{1}{3}$ *c. ft.*



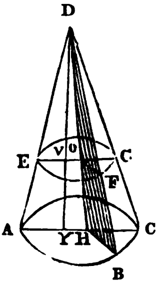
4. What is the capacity of a coal waggon, whose inside dimensions are as follows; at the top the length and breadth are 81·5 and 55 in., the corresponding dimensions at the bottom are 41 and 29·5 in., and the perpendicular depth 47·25 in.? *Ans.* 126340 *c. in.*

5. What is the solidity of the frustum ABCGE of a cone, when the diameter AC of the base $=4$ ft., that of the top EG $=2$ ft., and the altitude VY $=18$ ft.?

By the general rule. Area base $=4^2\times{\cdot}7854$, area top $=2^2\times{\cdot}7854$, area middle section $=3^2\times{\cdot}7854$; $\therefore$ solidity frustum

$=\frac{1}{6}\times 18(4^2+2^2+4\times 3^2){\cdot}7854=131{\cdot}947$ *c. ft.*

Or thus without the rule. Let ABCD be the complete cone; then in the similar triangles ACD and EGD, because the base AC ($=4$) is double the base EG ($=2$), the perpendicular DY will be double the perpendicular DV; therefore DV $=$ VY $=18$, and DY $=36$; *hence we have,*



$$\text{Solidity cone ABCD} = \tfrac{1}{3} \times 4^2 \times {\cdot}7854 \times 36 = {\cdot}7854 \times 192,$$

$$\text{,, ,, EGD} = \tfrac{1}{3} \times 2^2 \times {\cdot}7854 \times 18 = {\cdot}7854 \times 24;$$

$$\therefore \text{ Solidity frustum} = \text{cone ABCD} - \text{cone EGD}$$

$$= {\cdot}7854\,(192 - 24) = 131{\cdot}947 \; c.ft.$$

Let us now solve this question generally; for this purpose put $\text{D} = \text{AC}$, $d = \text{EG}$, $h = \text{VY}$, and $x = \text{DV}$, then from the similar triangles ACD and EGD, we have,

$$\text{AC} : \text{DY} :: \text{EG} : \text{DV},$$

$$\text{or } \text{D} : x + h :: d : x, \quad \therefore \; x = \frac{hd}{\text{D} - d},$$

and $\therefore x + h$, or $\text{DY} = \dfrac{h\text{D}}{\text{D} - d}$; hence we now have,

$$\text{Solidity cone ABCD} = {\cdot}7854 \times \text{D}^2 \times \frac{h\text{D}}{\text{D} - d} \times \tfrac{1}{3} = \frac{{\cdot}2618\,h}{\text{D} - d} \times \text{D}^3.$$

$$\text{,, ,, EGD} = {\cdot}7854 \times d^2 \times \frac{hd}{\text{D} - d} \times \tfrac{1}{3} = \frac{{\cdot}2618\,h}{\text{D} - d} \times d^3.$$

These expressions show that the solidity of similar cones are as the cubes of their diameters;

$$\therefore \text{ Solidity frustum ABCGE} = \frac{{\cdot}2618\,h}{\text{D} - d}\,(\text{D}^3 - d^3);$$

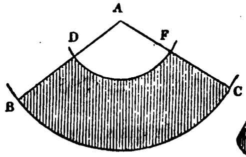
which is the analytical expression of the rule commonly given for the solidity of a frustum of a cone. The general rule for the solidity of a prismoid may be readily reduced to this formula.

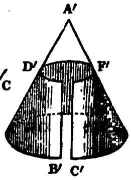
6. What is the solidity of the frustum of a cone, the diameter of the greater end being 5 ft., that of the less end 3 ft., and the perpendicular height 4·5 ft.?

Ans. 57·7269 *c. ft.*

7. What is the convex surface of the frustum of an upright cone, the circumference of the base being 30 in., that of the top 20 in., and the length of the slant side 40 in.?

Convex surface $= \tfrac{1}{2}(30 + 20) \times 40 = 1000$ sq. in.





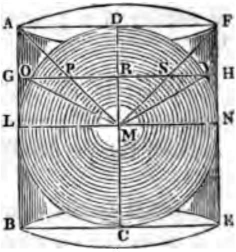
To render the rule in this case apparent; let the sector ABC be cut out of a piece of paper, and folded, as in the second figure, by bringing the edges AB and AC together; then the sector ABC will form the cone A′B′C′, and the sectoral ring BCFD will form the frustum B′C′F′D′. Hence the convex surface of the cone A′B′C′ is equal to the area of the sector ABC, or $\frac{1}{2} \times$ AB $\times$ arc BC, that is, half the slant height of the cone × circum. base; and the convex surface of the frustum B′C′F′D′ is equal to the area of the sectoral ring BCFD, or $\frac{1}{2}$(arc BC + arc DF) × BD, that is, half the sum of the perimeters of the two ends of the frustum × the slant height.

8. Required the convex surface of the frustum of a right cone, the diameters of the ends being 16 and 8 ft., and the slant height 20 ft. *Ans.* 753·984 *sq. ft.*

23. Problem. To find the solidity of a sphere.

Rule. Take two-thirds of the solidity of the circumscribing cylinder, and it will be the solidity of the sphere. (Geo. Art. **93.**) Or multiply the cube of the diameter of the sphere by ·5236, and the product will be the solidity.

Examples.



1. What is the solidity of the sphere CLDN, whose diameter DC or LN is 50 in.?

Here the solidity of the sphere will be $\frac{2}{3}$ of the solidity of the circumscribing cylinder ABEF;

$\therefore$ Solidity cylinder $= 50^2 \times {\cdot}7854 \times 50 = 50^3 \times {\cdot}7854$,

$\therefore$ *Solidity* sphere $= 50^3 \times {\cdot}7854 \times \frac{2}{3} = 50^3 \times {\cdot}5236 = 65450$ *c.in.*

2. What is the solidity of a sphere, whose diameter is 2 ft. 8 in.? *Ans.* 9·929 *c. ft.*

3. What is the solidity of a sphere, whose radius is 5 ft.? *Ans.* 523·6 *c. ft.*

4. The diameter of the moon is 2180 miles, what is her solidity? *Ans.* 5424617475 *c. miles.*

5. Required the number of cubic feet of material in a hollow globe, whose exterior diameter is 9 ft., and the thickness 2 in.

Here the interior diam. $= 9 - \frac{1}{3} = \frac{26}{3}$ ft.

$$\text{Solidity of the exterior sphere} = 9^3 \times {\cdot}5236,$$

$$\text{,,} \quad \text{,,} \quad \text{interior} \quad \text{,,} \quad = (\tfrac{26}{3})^3 \times {\cdot}5236,$$

$$\therefore \text{ Solidity of the shell} = \{9^3 - (\tfrac{26}{3})^3\}\, {\cdot}5236 = 40{\cdot}86.$$

6. The exterior diameter of a hollow globe of metal is 10 in., and the interior diameter 8 in.; required the number of cubic feet of metal, and also its weight, allowing 7200 oz. to each cubic foot. *Ans.* ·14786 *c. ft.; and* 1064·6 *oz.*

24. Problem. To find the surface of a sphere, or any spherical segment.

Rule. 1. Find the convex surface of the circumscribing cylinder, and it will be the surface of the sphere. Or multiply the square of the diameter of the sphere by 3·1416, and the product will be its surface. 2. The convex surface of any spherical segment is equal to the convex surface of the corresponding portion of the circumscribing cylinder.

Examples.

1. Required the surface of a sphere LDNC, whose diameter LN or DC is 7 ft. (See last figure.)

Here, the circum. of the base of the cylinder $= 3{\cdot}1416 \times 7$;

$\therefore$ Convex surface cylinder or sphere

$$= 3{\cdot}1416 \times 7 \times 7 = 153{\cdot}9384.$$

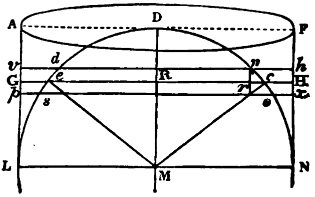
2. Required the convex surface of the segment VDOR, of the sphere whose diameter LN or DC is 6 ft., the height DR of the segment being 2 ft. (See last figure.)

Here the convex surface of the segment VDO will be equal to the convex surface of the corresponding portion HFAG of the cylinder.

$\therefore$ Circum. cylinder $= 3{\cdot}1416 \times 6$;

$\therefore$ convex surface cy. HFAG or seg. VDO $= 3{\cdot}1416 \times 6 \times 2$
$= 37{\cdot}6992$.

In order to establish the general rule; let $vdnh$ and $psox$ be two sections, of the sphere and cylinder, parallel to AF or LN, and very near to each other, and draw nr perpendicular to so; then if the circle LDN &c., and the square LAFN &c., be made to revolve upon the axis DM, the arc no will describe a portion of the surface of the sphere, and hx a portion of the surface of the cylinder. As nd and os are taken indefinitely near to each other, no may be regarded as a straight line, and the surface described by it as the surface of a frustum of a cone; therefore, taking ec as the diameter of the mean section, the spherical surface described by $no = 3{\cdot}1416 \times ec \times no$; and the corresponding surface of the cylinder, or the portion described by $hx = 3{\cdot}1416 \times \text{GH} \times nr$.



Now from the similar triangles $\text{MR}c$ and nro, we have,

$$\text{M}c : \text{R}c :: no : nr, \quad \therefore \text{M}c \times nr = \text{R}c \times no;$$

$$\therefore 2\text{M}c \times nr = 2\text{R}c \times no, \text{ or } \text{GH} \times nr = ec \times no.$$

Hence the convex surface of the sphere is equal to the corresponding convex surface of the cylinder; and as this is true for every section which we can make, it follows, that the whole surface of the sphere is equal to the whole convex surface of the cylinder; and moreover the convex surface of any spherical segment eRcD is equal to the convex surface of the corresponding portion GHFA of the cylinder.

3. What is the surface of a globe, whose diameter is 2 ft. 10 in.? *Ans.* 25·22 *sq. ft.*

4. The diameter of a sphere is 1 ft. 9 in.; required the convex surface of the segment whose height is 9 in. *Ans.* 593·76 *sq. in.*

5. If the mean diameter of the earth be 7912 miles, what will be its surface? *Ans.* 196662895 *sq. miles.*

6. The circumference of a ball is 15·708 inches; required its surface. *Ans.* 78·54 *sq. in.*

25. Problem. To find the solidity of a segment of a sphere.

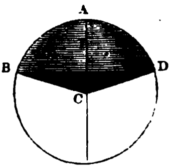
Rule. From three times the diameter of the sphere, subtract twice the height of the segment; multiply the remainder by the square of the height, and that product again by ·5236, will give the solidity.

Or thus. To three times the square of the radius of the base of the segment add the square of the height, and this sum multiplied by the height and ·5236 will give the solidity.

Examples.

1. Required the solidity of the spherical segment ABD, whose height AI is 2, and the radius CA or CB of the sphere is 5.

Here the segment ABD is equal to the spherical sector ABCD minus the cone BDC.



Now the spherical sector ABCD may be regarded as being made up of an indefinite number of little cones, the bases of which form the surface of the segment, and all the vertices meet in the centre C of the sphere; hence we have,

Surface segment ABD $= 3·1416 \times 10 \times 2$,

∴ *Solidity sector* ABCD $= 3·1416 \times 10 \times 2 \times \frac{5}{3} = 104·72$.

In the right-angled triangle BCI we have, $\text{CI} = 5 - 2 = 3$, $\text{BI}^2 = 5^2 - 3^2 = 16$, and $\therefore$ $\text{BD}^2 = 16 \times 4 = 64$,

$$\therefore \text{ Solidity cone BDC} = 64 \times {\cdot}7854 \times \tfrac{3}{3} = 50{\cdot}2656;$$

$$\therefore \text{ Solidity segment ABD} = 104{\cdot}72 - 50{\cdot}2656 = 54{\cdot}4544.$$

By the rule.

$$\text{Solidity} = {\cdot}5236 \times 2^2 (3 \times 10 - 2 \times 2) = 54{\cdot}4544.$$

Let us now investigate this problem generally; for this purpose, put D = the diameter of the sphere, $h = \text{AI}$, the height of the segment, and $r = \text{BI}$, the radius of the base of the segment; then

$$\text{Surface segment ABD} = 3{\cdot}1416\,\text{D}h,$$

$$\therefore \text{ Solidity sector ABCD} = 3{\cdot}1416\,\text{D}h \times \frac{\text{D}}{2} \times \tfrac{1}{3} = {\cdot}5236\,\text{D}^2 h.$$

Now in the right-angled triangle BCI, we have, $\text{CI} = \frac{\text{D}}{2} - h$,

$$\text{BI}^2 = \left(\frac{\text{D}}{2}\right)^2 - \left(\frac{\text{D}}{2} - h\right)^2 = \text{D}h - h^2, \text{ and } \therefore \text{BD}^2 = 4(\text{D}h - h^2);$$

$$\therefore \text{ Solidity cone BDC} = \text{BD}^2 \times {\cdot}7854 \times \text{CI} \times \tfrac{1}{3}$$
$$= {\cdot}5236 (\text{D}h - h^2)(\text{D} - 2h);$$

$\therefore$ Solidity segment ABD

$$= {\cdot}5236\,\text{D}^2 h - {\cdot}5236(\text{D}h - h^2)(\text{D} - 2h) = {\cdot}5236\,h^2(3\text{D} - 2h);$$

which is the first rule given.

In order to eliminate D from this expression, we have already found BI^2 or $r^2 = \text{D}h - h^2$, $\therefore$ $\text{D} = \dfrac{r^2 + h^2}{h}$. Substituting this value for D, we have, after a little reduction,

$$\text{Solidity segment ABD} = {\cdot}5236\,h(3r^2 + h^2);$$

which is the second rule given.

2. The diameter of a sphere is 12 ft.; what is the solidity of a segment whose height is 3 ft.? *Ans.* 141·372 *c. ft.*

3. What is the solidity of the segment of a sphere, the radius of whose base is 20, and its height 10? *Ans.* 6806·8.

4. The diameter of a sphere is 40; what is the solidity of a segment whose height is 5? *Ans.* 1439·9.

TO FIND THE CONTENT OF IRREGULAR SOLIDS.

26. Let the irregular solid be divided into any number of parts by equidistant sections; put $a_1, a_2, a_3, \ldots a_n$, for the areas of these equidistant sections, and $2d$ for their common distance apart; moreover, put $m_1, m_2, m_3, \ldots m_{n-1}$, for the areas of the middle sections, taken in order; then by Art. **22.** we have,

$$\text{content 1st part} = \frac{2d}{6}(a_1 + a_2 + 4m_1),$$

$$\text{content 2nd part} = \frac{2d}{6}(a_2 + a_3 + 4m_2),$$

$$\text{content 3rd part} = \frac{2d}{6}(a_3 + a_4 + 4m_3),$$

* * * *

$$\text{content last part} = \frac{2d}{6}(a_{n-1} + a_n + 4m_{n-1}).$$

Adding these together, we find the total content to be,

$$\frac{d}{3}\left\{a_1 + a_n + 4(m_1 + m_2 + \ldots + m_{n-1}) \right.$$
$$\left. + 2(a_2 + a_3 + \ldots + a_{n-1})\right\}$$

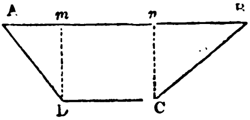
This formula may be expressed in language as follows:—

To the sum of the extreme sections add four times the sum of the middle or mean sections, and two times the sum of the given intermediate sections; then this sum, multiplied by one-third the common distance between any section and the next middle section, will give the content of the solid.

This expression is a generalisation of Thomas Simpson's celebrated formula for finding the area of an irregular curvilineal space.

27. *To find the content of drains, &c.*

1. Required the content of a drain whose cross sections, taken at 30 ft. apart, are as follows: the top breadths AB are, 4, 3, and 5 ft.; the correspondnig depths Dm or Cn are, 3, 2, and 4 ft., and the breadth DC at the bottom 2 ft., being the same for every section of the drain.



Here the sections are all trapezoids; hence we have,

area 1st section, or $a_1 = \frac{1}{2}(4+2) \times 3 = 9$,
area 2nd section, or $a_2 = \frac{1}{2}(3+2) \times 2 = 5$,
area 3rd section, or $a_3 = \frac{1}{2}(5+2) \times 4 = 14$;

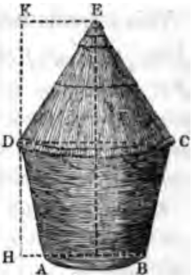
area 1st middle section, or $m_1 = \frac{1}{2}\left(\frac{4+3}{2} + 2\right) \times \frac{3+2}{2} = 6\frac{7}{8}$

area 2nd middle section, or $m_2 = \frac{1}{2}\left(\frac{3+5}{2} + 2\right) \times \frac{2+4}{2} = 9.$

Hence we have, by the general formula, observing that $2d = 30$, and $\therefore\ d = 15$,

$$\text{Content} = \frac{15}{3}\left\{9 + 14 + 4(6\tfrac{7}{8} + 9) + 2 \times 5\right\} = 482{\cdot}5 \text{ c. ft.}$$

2. Let the annexed figure represent a hay-stack, where the circumference at AB = 40 ft.; at the eaves DC = 60 ft.; and the height HD = DK = 15 ft. Required the solid eet in the stack.



Here, by Art. 9., we find the area of the section AB, or $a_1 = 128$ nearly; the area of the section DC, or $a_2 = 288$ nearly; and the section at E, or $a_3 = 0$. The circumference of the middle section between AB and DC will be $\frac{1}{2}(40 + 60) = 50$; and between DC and E will be $\frac{1}{2}(60 + 0) = 30$; hence we find $m_1 = 200$ nearly, and $m_2 = 72$ *nearly.*

$$\therefore \text{Content} = \frac{7\frac{1}{2}}{3}\left\{128 + 0 + 4\,(200 + 72) + 2 \times 288\right\}$$
$$= 4480 \text{ c. ft.}$$

The solidity, in this case, may be found by adding the content of the frustum ABCD to the content of the cone DCE.

28. *To find the content of railway cuttings and embankments.*

Let ABCD (see 1st fig. of Art. **27.**) represent the cross section of a railway cutting, where DC is supposed to be the breadth of the rail, CB and DA the sloping sides, Dm or Cn the perpendicular height, and in this case nB $=$ mA the distance which the sides fall away from the vertical lines Cn or Dm. The slope of the sides is usually expressed by the distance which they fall away for every vertical foot. An embankment is merely a cutting in a reversed position.

Let it be required, for example, to find the area of a section, when the breadth of the rail is 30 ft., the vertical height 8 ft., and the slopes 3 to 1.

Here, as the side falls away 3 ft. in 1 vertical foot, in 8 vertical feet it will fall away 8 times 3 ft., or 24 feet, that is, $n\text{B} = m\text{A} = 24$ ft.; $\therefore$ $\text{AB} = mn + 2\,n\text{B} = 30 + 2 \times 24 =$ 78 ft.; and the area of the trapezoid $\text{ABCD} = \frac{1}{2}(30 + 78) \times 8$ $= 432$ sq. ft.

In general, let $\text{DC} = g$, $\text{C}n = h_1$, and the slopes p to 1; then $n\text{B} = h_1$ times $p = p\,h_1$; $\text{AB} = g + 2p\,h_1$; and $\therefore$ area $\text{ABCD} = \frac{1}{2}(\text{DC} + \text{AB}) \times \text{C}n = (g + p\,h_1)\,h_1$.

In the next, or 2nd section, let $h_2 =$ the height, the other dimensions being the same; then the area of this section $= (g + p\,h_2)\,h_2$; in like manner the area of the 3rd section $= (g + p\,h_3)\,h_3$, and so on.

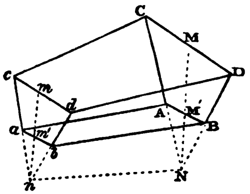
To find the area of the middle section between the 1st and 2nd sections, we have merely to substitute the mean of h_1 and h_2, or $\frac{1}{2}(h_1 + h_2)$ for the height; thus we have,

$$\{g + p \times \tfrac{1}{2}(h_1 + h_2)\}\,\tfrac{1}{2}(h_1 + h_2) \text{ for the area;}$$

and so on to any other middle section.

In the general formula, of Art. **26.**, for the content of a cutting or embankment, a_1, a_2, a_3, &c., stand for the 1st, 2nd, 3rd, &c., sections just given; m_1, m_2, &c., for the 1st, 2nd, &c., middle sections; and d for one half the common distance between the sections.

1. Let ABC*dab* be a cutting, where the ground CD*dc* is a plane parallel to the plane of the rail AB*ba*. The breadth of the rail *ab* or AB is 30 ft., the length B*b* 1 chain or 66 ft., the vertical height MM′ 8 ft., *mm′* 6 ft., and the slopes $1\frac{1}{2}$ to 1. Required the solidity of the cutting.



Here $\text{CD} = 30 + 1\frac{1}{2} \times 8 \times 2 = 54$, and $\therefore$ area $\text{ABDC} = \frac{1}{2}(30 + 54)\,8 = 336$, or we may more readily obtain this result from the preceding formula by putting 30 for g, $1\frac{1}{2}$ for p, and 8 for h_1. Similarly we find area $abdc = 234$. To find the area of the middle section, we have the mean height $= \frac{1}{2}(8 + 6) = 7$, the mean top width $= 30 + 1\frac{1}{2} \times 7 \times 2 = 51$,

$$\therefore \text{ the middle area} = \tfrac{1}{2}\,(30 + 51)\,7 = 283{\cdot}5.$$

Hence we have by the prismoidal formula, Art. **22.**, Solidity cutting $= \frac{66}{6}\,\{336 + 234 + 4 \times 283{\cdot}5\} = 18744$ c. ft.

2. Required the solid content of a cutting or embankment, whose heights taken at 2 chains apart, are 0, 10, 30, 40, and 0 ft., the width of the rail 30 ft., and the slopes 4 to 1.

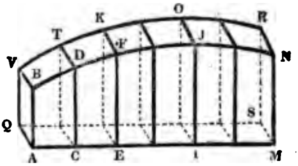
Here $d = \frac{1}{2}$ of 2 chains $= 66$ ft.; $h_1 = 0$, $h_2 = 10$, $h_3 = 30$ ft., &c.; the height of the 1st middle section $= \frac{1}{2}(0 + 10) = 5$, the 2nd $= \frac{1}{2}(10 + 30) = 20$, the 3rd $= \frac{1}{2}(30 + 40) = 35$, and the 4th $= \frac{1}{2}(40 + 0) = 20$; hence we have, by the general formula for the areas of the sections, $a_1 = 0$; $a_2 = (30 + 4 \times 10)\,10 = 700$; $a_3 = (30 + 4 \times 30)\,30 = 4500$; $a_4 = (30 + 4 \times 40)\,40 = 7600$; and $a_5 = 0$; $m_1 = (30 + 4 \times 5)\,5 = 250$; $m_2 = (30 + 4 \times 20)\,20 = 2200$; $m_3 =$

$(30 + 4 \times 35)35 = 5950$; and $m_4 = (30 + 4 \times 20)20 = 2200$. Therefore by the formula, Art. **26.**, we find the solidity to be,

$$\tfrac{66}{3}\{0 + 0 + 4\,(250 + 2200 + 5950 + 2200) + 2\,(700 + 4500 + 7600)\} = 1496000 \text{ c. ft.}$$

3. Required the solidity of a cutting, whose heights taken at 2 chains apart are, 8 and 10 ft., the width of the rail 32 ft., and the slopes 2 to 1. *Ans.* 59488 *c. ft.*

For further examples see Tate's Mechanics. A more general formula for cuttings will be hereafter investigated.



29. Let AQRM be a solid bounded on one side at least by the curve surface BVNR; and let AV, CT, EK, &c., be equidistant sections perpendicular to AM. Now if these sections be taken near to each other, BDF will be nearly a straight line, and CT may be taken, without any sensible error, as the mean or middle sections of AV and EK, and so on to the other sections. Hence the general formula of Art. **26.** will express the content of any irregular solid very nearly. In the present case d is put for AC, the common distance between the sections; a_1 and a_4 for the *extreme* sections AV and MR respectively; m_1, m_2, &c., the *even* sections CT, &c.; and a_2, a_3, &c., the *odd* sections EK, IO, &c. Thus we have,

$$\text{Solidity AQRM} = \frac{d}{3}\{a_1 + a_4 + 4(m_1 + m_2 + m_3) + 2\,(a_2 + a_3)\}.$$

This formula may be expressed in language as follows:—

To the sum of the extreme sections, add four times the sum of the even sections and two times the sum of the odd sections; then this sum, multiplied by one third the common distance between the sections, will give the content of the solid nearly. The number of parts into which the solid is divided should always be *even*.

For example, let it be required to find the solidity of an irregular solid, where the equidistant sections, taken at 9 ft. apart, are in order as follows:—

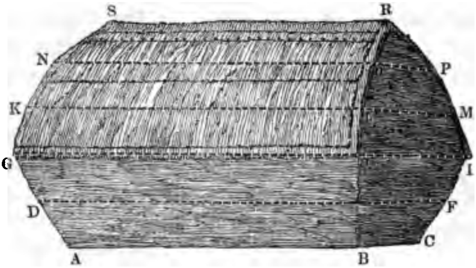
5, 6, 14, 17, 20, 12, and 9 square feet.

Here sum of the *extreme* areas $= 5 + 9 = 14$,
sum of the *even* areas $= 6 + 17 + 12 = 35$,
sum of the *odd* areas $= 14 + 20 = 34$;

$\therefore$ Solidity $= \frac{9}{3}\{14 + 4 \times 35 + 2 \times 34\} = 666$ c. ft.

30. *To find the content of a hay-stack.*

Let the annexed figure represent a hay-stack, whose horizontal sections are rectangles. Between ABC and NOP the dimensions of the equidistant sections, taken at 6 ft. apart, are as follows: AB = 40, BC = 14, DE = 42, EF = 18, GH = 46, HI = 23, KL = 45, LM = 20, NO = 44, OP = 14 feet. The perpendicular height of SR from NOP is 5 ft., and SR = 40 ft. Required the number of cubic feet in the stack.

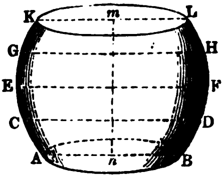


Here we find the areas of the equidistant sections to be in their order, 560, 756, 1058, 900, and 616.

$\therefore$ Content ANPB $= \frac{6}{3}\{560+616+4(756+900)+2 \times 1058\}$
$= 19832$ c. ft.

The top part NOPS, calculated by the prismoidal formula, will be found to be 1493 c. ft.; therefore we find the total content to be 21325 c. ft.

31. *To gauge a cask.*



Let the annexed figure represent a vessel whose horizontal sections are circles. The equidistant diameters, taken at 12 inches apart, are as follows; AB = 50, CD = 60, EF = 70, GH = 60, and KL = 54 inches. What is its content in imperial gallons?

Content in c. in.

$$= \tfrac{12}{3} \times \cdot 7854 \{50^2 + 54^2 + 4(60^2 + 60^2) + 2 \times 70^2\}.$$

Dividing this result by 277·274, we find the number of gallons to be 499 nearly.

32. The general formula of Art. **26.** will hold true for the area of a curvilineal space ABNM, by merely changing the equidistant areas into equidistant ordinates. (See fig. to Art. **29.**)

Let the face QVRS be parallel to ABNM, and AQSM perpendicular to it. Put b for the uniform breadth AQ, and p_1, p_2, p_3, &c., for the equidistant ordinates AB, CD, EF, &c.; then the content of this solid will be equal to the area of ABDNM multiplied by b. But $a_1 = p_1 b$, $m_1 = p_2 b$, $a_2 = p_3 b$, $m_2 = p_4 b$, $a_3 = p_5 b$, and so on; hence we have by equality and substitution, area ABNM × b

$$= \frac{d}{3}\{p_1 b + p_7 b + 4(p_2 b + p_4 b + p_6 b) + 2(p_3 b + p_5 b)\}.$$

Dividing by b, we have,

$$\text{area ABNM} = \frac{d}{3}\{p_1 + p_7 + 4(p_2 + p_4 + p_6) + 2(p_3 + p_5)\}$$

which is the usual form of Thomas Simpson's rule. From the demonstration here given, it appears that this formula is merely a particular case of the one given in Art. **26.**

Let it be required, for example, to find the area of a curvilineal space, whose extreme ordinates are 8 and 9 ft.

respectively, the intermediate equidistant ordinates, taken in order, are 9, 11, 15, 16, and 14 ft.; and the common distance between them 5 ft.

Here the sum of the extreme ordinates $= 8 + 9 = 17$.
The sum of the even ordinates $= 9 + 15 + 14 = 38$.
The sum of the odd ordinates $= 11 + 16 = 27$.

$\therefore$ Area $= \frac{5}{3}(17 + 4 \times 38 + 2 \times 27) = 371\frac{2}{3}$ sq. ft.

SPECIFIC GRAVITY.

33. The specific gravity of a body is its weight as compared with an equal bulk of pure water. As the weight of a cubic foot of water, at the temperature of 40°, is 1000 oz., it is customary to consider the specific gravity of a body as the weight of a cubic foot of it. The following table contains the weight of a cubic foot of the various kinds of material in ounces.

Table of Specific Gravities.

Metals, Platinum	21470	Clay	2160
— Gold	19250	Brick	2000
— Mercury	13580	Light earth	1984
— Lead	11352	Chalk	from 2252 to 2657
— Silver	10470	Sand	1520
— Copper	8900	Coal	from 1020 to 1300
— Cast Brass	8399	Sea water	1028
— Steel	7840	Pure water	1000
— Iron	7700	Woods, Mahogany	1063
— Cast Iron	7066	— Oak	934
— Tin	7299	— Beech	690
Stones, Granite	9956	— Ash	760
— Marble	2700	— Fir	553
— Bristol	2640	— Cork	240
— Portland	2496	Atmospheric air	1·2
Glass, flint	3000		

34. *To find the weight of a body from its magnitude.*

1. What is the weight of a block of marble 12 feet long, *3 ft. broad*, and 1 ft. 6 in. thick?

Content $= 12 \times 3 \times 1\frac{1}{2} = 54$ c. ft.
Wt. of 1 c. ft. $= 2700$ oz.

$\therefore$ Wt. of 54 c. ft. $=$ 54 times 2700 oz. $= 9112{\cdot}5$ lbs.

2. Required the weight of a beam of fir 20 ft. long, 9 in. deep, and 3 in. thick. *Ans.* 129·6 *lbs.*

3. A cast-iron pipe is 3 inches diameter in the bore, and $\frac{1}{2}$ inch in thickness; required the weight of a running foot.

Here, by Problem **11.**, we find the section of the metal to be 5·4978 sq. in.

$\therefore$ Content of 1 ft. of pipe $= 5{\cdot}4978 \times 1 \div 144$

$\therefore$ Wt. in lbs. $= 5{\cdot}4978 \times 7066 \div 144 \times 16 = 16{\cdot}86$.

4. What will be the weight of a running foot of leaden pipe, 2 inches diameter in the bore, and $\frac{1}{4}$ inch in thickness? *Ans.* 8·7 *lbs.*

35. *To find the magnitude of a body from its weight.*

1. A piece of beech weighs 300 lbs.; required its solidity.

Weight of 1 c. ft. $= 690$ oz.

$\therefore$ No. c. ft. $= 300 \times 16 \div 690 = 7$ c. ft. nearly.

2. Required the number of cubic feet in 1 ton of iron. *Ans.* 4·655.

3. If a horse can draw $1\frac{1}{2}$ tons, how many cubic feet of oak will he draw? *Ans.* 57·55.

4. How many cart-loads of clay will there be in a drain 90 ft. long, 3 ft. broad, and 2 ft. deep, allowing $1\frac{1}{4}$ tons for each load? *Ans.* 26·03.

FLOATING BODIES — TONNAGE OF VESSELS.

36. When a body floats in water, the weight of the fluid displaced is equal to the weight of the body. Thus, for example, if the weight of the floating body be 4000 oz., it would exactly displace 4 cubic feet of water. The tonnage of a vessel is the weight, in tons, necessary to sink it to a certain depth, or line of floatation. To find the tonnage of a vessel, therefore, we must calculate the volume of that portion of the vessel which is immersed; then the weight of water, in tons, equal to this volume, minus the weight of the vessel itself, will be the tonnage. The chief difficulty in this calculation consists in finding the volume of the solid immersed. The tonnage may be found in all cases, with a sufficient degree of precision, by the following method:— 1. Divide the length of the vessel into any convenient even number of equal parts, and make exact drawings of these sections. 2. Calculate the areas of these sections below the water line, by measuring off equidistant ordinates after the method explained in Art. **32.** 3. Calculate the volume of the solid of immersion in cubic feet by Art. **29.** 4. Multiply this volume by 62·5, the weight of a cubic foot of water in lbs., and divide this product by 2240, and it will give the number of tons in the whole floating mass; from which subtract the weight of the vessel in tons, and it will give the tonnage required.

For example, let the areas of the equidistant sections, taken in order, at every 9 ft. apart, be 0, 20, 60, 30, and 0 square ft.; and the weight of the vessel 5 tons. Required the tonnage.

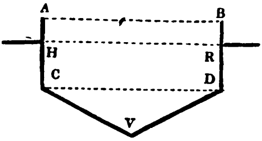
Vol. displaced fluid $= \frac{9}{3}\{4(20+30)+2\times 60\} = 960$ c. ft.

$\therefore$ Wt. displaced fluid $= 960 \times 62{\cdot}5 \div 2240 = 26{\cdot}7$ tons.

$\therefore$ The tonnage $= 26{\cdot}7 - 5 = 21{\cdot}7$.

The weight of the vessel may be ascertained, by adding together the weight of the material in the different parts of which it is composed; or by finding the weight of the fluid displaced when the vessel is floating without any load.

1. For the sake of simplicity, let us suppose that the vessel has the form of a prism, of which ACVDB is the section. Let the length $= 20$, $AB = CD = 8$, $AC = BD = 3$, $CV = DV = 5$ ft.; the weight of a superficial foot of the material in the sides 15 lbs.; and the weight of the other parts of the vessel 2 tons. Required the load, so that the water line HR may be 2 ft. from the top.



To find the weight of the water displaced.

Area $HCDR = HR \times HC = 8 \times 1 = 8$ sq. ft.

Area $CVD = \sqrt{9 \times 1 \times 4 \times 4} = 12$ sq. ft.

$\therefore$ Area $HRDVC = 8 + 12 = 20$ sq. ft.

$\therefore$ Vol. displaced water $= 20 \times 20 = 400$ c. ft.

$\therefore$ Wt. displaced water $= 62{\cdot}5 \times 400 = 25000$ lbs.

To find the weight of the vessel.

Girt of the vessel $= 2 \times 3 + 2 \times 5 = 16$ ft.

$\therefore$ Area sides and bottom $= 16 \times 20 = 320$ sup. ft.

Area ends $= (8 \times 3 + 12) \times 2 = 72$ sup. ft.

$\therefore$ Total surface $= 320 + 72 = 392$ sup. ft.

Wt. total surface $= 392 \times 15 = 5880$ lbs.

$\therefore$ Total wt. $= 5880 + 4480 = 10360$ lbs.

$\therefore$ Tonnage $= (25000 - 10360) \div 2240 = 6{\cdot}53$.

2. Required the same as in the last example, when $AB = 16$, $AC = 7$, $CV = 10$, $AH = 3$, and the length $= 30$ feet; the other things being the same. *Ans.* 82·77 *tons.*

PRINCIPLES OF ESTIMATION.

1. The difficulty which workmen find in estimating the value of work, does not so much proceed from their ignorance of the requisite rules of mensuration, as it does from their want of a proper knowledge of the principles and methods upon which correct estimation is based. As the wages of workmen, as well as the cost of material, is variable, "price-books" are never implicitly followed by an intelligent builder, even though he should reside in the locality where such books are published. In article **24.** are given methods for adapting a London price-book to any rate of wages, or price of material. While the rate of wages is variable, it may, however, be presumed that the quantity of any specified kind of work, which an average workman will execute in a day, is a constant quantity. This constant, which we shall call *the constant of labour*. may be the number of superficial feet, or decimal parts of a superficial foot, the number of squares, or decimal parts of squares, and so on, as the case may be, which an average workman will execute in one day. Having obtained the constant of labour for any particular kind of work, the estimation of the value of any specified amount of that work becomes a matter of pure arithmetic. These constants are especially valuable in carpenters and joiners' work. It will be hereafter explained how workmen may determine these constants for themselves.

The custom of estimating and contracting for work by the "lump" is little better than a modified form of gambling, by which the contractor may either make himself a fortune, to the injury of his employer, or render himself penniless, with all the moral evils resulting from hopeless and irretrievable pecuniary embarrassment. Teachers of elementary schools and Mechanics' Institutes would confer a *great benefit* on society, by teaching the simple and fundamental principles of estimation, rather than waste the time of their pupils in giving "sums," which are only calculated to amuse the curious, without conveying any satisfactory information to the man who is engaged in the business of life. It is true that some of these "sums" may exercise the mind, but it is an important fact, which should never be lost sight of, that those investigations which have the greatest practical bearing invariably form the most healthful and instructive exercise to the intellectual powers.

MASON WORK.

2. Mason work is usually estimated by the square yard or square foot. The cost of a yard of any particular kind of work depends upon the nature and value of the material, the rate of wages, and the expense of cartage. These, of course, vary with the locality where the work is done. The following examples will show how the workman should proceed in estimating, under the given conditions, the price of any kind of work. The experience of the workman will enable him to supply the proper *data* for the calculation in each case.

3. *Cutting foundations, &c.*

Required the cost in cutting the foundations for the two side walls, and two ends, of a building 190 ft. long, and 74 ft. broad, the cutting being 3 ft. wide and 2 ft. deep. The material is such as to require 3 pickmen to 2 shovellers. The shoveller can lift 500 c. ft. of earth per day, and can fill a barrow while a barrowman wheels a full barrow to the distance of a relay of 120 ft. and returns with the empty barrow. Allowing each labourer 2*s.* 6*d.* per day.

No. c. ft. in the cutting $= (190 \times 2 + 74 \times 2) \times 3 \times 2 = 3168$.

Then, in completing the excavation and transport, we find the number of day's work for each as follows:—

$$\text{Shovellers} = \frac{3168}{500} = 6{\cdot}336$$

$$\text{Barrowmen} = 6{\cdot}336$$

$$\text{Pickmen} = \tfrac{3}{2} \text{ of } 6{\cdot}336 = \underline{9{\cdot}504}$$

$$\text{Total no. days' work} = \underline{\underline{22{\cdot}176}}$$

$$\therefore \text{ Cost} = 2s.\ 6d. \times 22{\cdot}176 = £2\ 15s.\ 5\tfrac{1}{4}d.$$

4. *Estimate of a sq. yd. of rubble walling 2 ft. thick.*

1. A horse will draw 1 ton on the common road. A cubic foot of freestone walling weighs 150 lbs. A load of rubble stones costs *5d.* at the quarry. A man with a cart and horse will convey 8 loads of stones per day for 6*s.* Required the cost of the material and cartage for a sq. yd.

Wt. of a sq. yd. of wall $= 2 \times 9 \times 150$ lbs.

$$\therefore \text{ No. loads in a sq. yd. of wall} = \frac{2 \times 9 \times 150}{2240} = 1{\cdot}205,$$

$$\therefore \text{ Cost stones in a sq. yd. of wall} = 1{\cdot}205 \times 5d. = 6d.$$

$$\text{No. sq. yds. conveyed per day} = \frac{8}{1{\cdot}205}$$

$$\therefore \text{ Cost cartage per sq. yd.} = 6s. \div \frac{8}{1{\cdot}205} = 10\tfrac{3}{4}d.$$

2. Three masons with two labourers will build 12 sq. yds. per day. A mason receives 4*s.* per day, and a labourer 3*s.* Required the cost of the labour in building 1 sq. yd.

Wages of the men per day $= 18s.$

$$\therefore \text{ Cost of 12 sq. yds.} = 18s.$$

$$\therefore \text{ Cost of 1 sq. yd.} = \frac{18s.}{12} = 1s.\ 6d.$$

3. The cost of 1 load of lime with 3 loads of sand for mortar is 10*s.*; and this mortar will build 30 sq. yds. of wall. Required the cost of mortar for 1 sq. yd.

Cost of mortar for 30 sq. yds. = 10*s.*

$$\therefore \text{ Cost of mortar for 1 sq. yd.} = \frac{10s.}{30} = 4d.$$

Collecting the various items, we have,

	s.	*d.*
Cost of the material =	0	6
„ cartage =	0	10¾
„ labour =	1	6
„ mortar =	0	4
	3	2¾
Profit at 10 per cent =		3¾
Total charge per sq. yd. =	3	6½

Example.—If 1 sq. yd. of rubble walling cost 3*s.* 6*d.*, what will be the charge for building the side walls, gables, and gable tops, of a house 36 ft. long, 18 ft. broad, and 21 ft. from the foundation to the eaves, the height of the gable tops being 6 ft.?

No. sq. yds. in the side walls and gables,

$$= (2 \times 36 + 2 \times 18) \times 21 \div 9 = 252,$$

No. sq. yds. in the gable tops $= 18 \times 6 \div 9 = 12$,

Total no. sq. yds. $= 252 + 12 = 264$,

$$\therefore \text{ Cost} = 3s.\ 6d. \times 264 = £46\ 4s.$$

5. *Estimate of a sq. yd. of ashler walling 2 ft. thick.*

1. The ashlers on an average are 9 inches in the bed, and cost 3*d.* per cubic foot at the quarry. The rubble stones for backing the ashlers are to be estimated at the same rate as in the preceding article. Required the cost of the stones in a sq. yd. of the wall.

$$\text{No. c. ft. of ashlers in 1 sup. yd.} = \frac{9}{12} \times 9 = 6{\cdot}75.$$

$$\therefore \text{ Cost of } „ \quad „ = 6{\cdot}75 \times 3d. = 1s.\ 8\tfrac{1}{4}d.$$

In this case the thickness of the rubble work $= 24 - 9 = 15$ inches; hence by Art. 4. we have,

Cost rubble stones per sup. yd. 24 in. thick $= 6d.$

$$\therefore \quad \text{,,} \quad \text{,,} \quad 15 \text{ in. thick} = \frac{6d. \times 15}{24} = 3\tfrac{3}{4}d.$$

$\therefore$ Total cost stones for 1 sq. yd. $= 1s.\ 8\tfrac{1}{4}d. + 3\tfrac{3}{4}d. = 2s.$

2. A man will hew, according to the style mentioned in the specification, 18 sq. ft. of ashler front per day. He receives 4*s.* per day, and blunts 16 tools at $\tfrac{1}{4}d.$ each sharping. Required the cost per sq. yd. in hewing the ashlers.

Cost of labour in 18 sq. ft. $= 4s.$

$$\therefore \quad \text{,,} \quad \text{,,} \quad 1 \text{ sq. yd.} = \frac{4s.}{2} = 2s.$$

$$\text{Cost of sharping tools per sq. yd.} = \frac{4d.}{2} = 2d.$$

$\therefore$ Total cost in hewing 1 sq. yd. $= 2s.\ 2d.$

Collecting these items, the cost of building, cartage, and mortar, being the same as in Art. 4., we have,

	s.	*d.*
Cost material =	2	0
„ cartage =	0	$10\frac{3}{4}$
„ hewing =	2	2
„ building =	1	6
„ mortar =	0	4
	6	$10\frac{3}{4}$
Profit at 10 per cent =		$8\frac{1}{4}$
Total charge for 1 sq. yd. =	7	7

EXAMPLE.—Required the charge for building an ashler wall 18 ft. long, and 9 ft. high, at 7*s.* 7*d.* per sq. yd.?

Ans. £6 16*s.* 6*d.*

6. *Estimate of a running foot of cornice.*

1. A cornice is 9 in. thick, 3 ft. in the breadth of the bed, and 16 inches girt in the moulding. A man can cut $3\frac{2}{3}$ sup. ft. of moulding in a day. The cost of the stone is 6*d.* per c. ft., the cost of cartage, wages, &c., the same as in Art. 4. Required the charge for a running foot?

No. c. ft. of stone in 1 run. ft. $= \frac{9}{12} \times 3 = 2\frac{1}{4}$.

$\therefore$ Cost of stones in 1 run. ft. $= 6d. \times 2\frac{1}{4} = 1s.\ 1\frac{1}{2}d.$

No. days to cut 1 sup. ft. $= 1 \div 3\frac{2}{3} = \cdot 272$,

which is the constant of labour in this particular case.

$\therefore$ Cost of labour and tools per sup. ft. $= 4s.\ 4d. \times \cdot 272 = 1s.\ 2d.$

$\therefore$ Cost of labour and tools for 1 run. ft. $= 1s.\ 2d. \times \frac{16}{12} = 1s.\ 6\frac{3}{4}d.$

Cartage per day $= 8 \times 2240$ lbs.

$\therefore$ Cartage for $2\frac{1}{4}$ c. ft., or 1 run. ft. $= \dfrac{150 \times 2\frac{1}{4}}{8 \times 2240} = \cdot 0188$ days

$\therefore$ Cost cartage per 1 run. ft. $\Rightarrow 6s. \times \cdot 0188 = 1\frac{1}{4}d.$

Collecting these items, we have,

	s.	*d.*
Cost material =	1	$1\frac{1}{2}$
„ labour =	1	$6\frac{3}{4}$
„ cartage =	0	$1\frac{1}{4}$
	2	$9\frac{1}{2}$
Profit at 10 per cent. =	0	$3\frac{1}{4}$
Total cost of 1 run. ft. =	3	$0\frac{3}{4}$

EXAMPLE.—Required the charge for a cornice 36 ft. long, at 3*s.* per running foot? *Ans.* £5 8*s.*

7. *Estimate of a sq. yd. of flagging, or chiselled pavement.*

1. The stones at the quarry cost 4*s.* per load of 1 ton, and average 4 in. thick. A man will dress and square up

12 sq. ft. per day, and 1 mason with 2 labourers will lay down 8 sq. yds. per day. A mason receives 3*s.* per day, and a labourer 2*s.*, and the cartage is at the same rate as in Art. 4. Required the charge for a sup. yd.

To find the value of 1 sq. yd. of the material.

No. c. ft. in 1 sq. yd. $= 9 \times \frac{4}{12} = 3$,

No. c. ft. in 1 load $= \frac{2240}{150} = 14{\cdot}93$,

$\therefore$ No. sup. yds. in 1 load $= \frac{14{\cdot}93}{3} = 4{\cdot}97$,

$\therefore$ Cost of a sup. yd. $= 4s. \div 4{\cdot}97 = 9\frac{3}{4}d.$

To find the cartage of 1 sup. yd.

No. sup. yds. in 8 loads $= 4{\cdot}97 \times 8 = 39{\cdot}76$,

$\therefore$ Cost of 39·76 sup. yds. $= 6s.$

$\therefore$ " 1 sup. yd. $= 6s. \div 39{\cdot}76 = 1\frac{3}{4}d.$

To find the labour in dressing 1 sq. yd.

Cost of 12 sq. ft. $= 3s.$

$\therefore$ " 1 sq. yd. $= 2s.\ 3d.$

Tools sharping for 12 sq. ft. $= 4d.$

" " 1 sq. yd. $= 3d.$

To find the labour in laying 1 sq. yd.

Cost of 8 sq. yds. $= 7s.$

$\therefore$ " 1 sq. yd. $= 7s. \div 8 = 10\frac{1}{2}d.$

Collecting the various items, we have,

	s.	*d.*
Cost of stones =	0	9¾
" cartage =	0	1¾
" labour dressing =	2	3
" sharping tools =	0	3
" labour laying =	0	10½
	4	4
Profit at 10 per cent. =	0	5¼
Total charge for 1 sq. yd. =	4	9¼

EXAMPLE.—Required the charge for flagging a kitchen floor 18 ft. by 15 ft., at 4*s.* 10*d.* per sq. yd. *Ans.* £7 5*s*

8. These calculations must only be regarded as *illustrations of the method*, by which the workman should proceed in forming estimates of any given kind of work. Every new case will require a distinct analysis of the various elements of cost.

BRICKLAYERS' WORK.

9. Brickwork is usually estimated by the rod, which contains 272 feet of surface, and with a standard thickness of a brick and half or $13\frac{1}{2}$ inches. If a wall has any other thickness, its content, reduced to the standard thickness, may be found by multiplying the superficial content of the wall, by the number of half bricks in its thickness, and dividing by 3. The number of cubic feet in a rod of standard thickness will be equal to $272 \times 13\frac{1}{2} \div 12 = 306$. Making a due allowance for the joints of the work, a rod of standard thickness will contain 4500 bricks. From these data the number of bricks in a given surface may be readily found: thus, for example, the number of bricks in 9 sq. ft. of wall 1 brick in thickness $= \frac{4500 \times 9}{272} \times \frac{2}{3} = 99$.

10. *Estimate of a rod of brickwork of standard thickness.*

1. The bricks cost 40*s.* per thousand at the kiln. A ton load costs 10*d.* in cartage. A sup. ft. of brickwork of standard thickness weighs 120 lbs. Two bricklayers with three labourers will build $\frac{1}{2}$ rod per day. A bricklayer receives 4*s.* per day, and a labourer 3*s.*

To find the cost of the bricks per rod.

$$\text{No. rods built by } 1000 = \frac{1000}{4500} = \frac{2}{9}.$$

$$\therefore \text{ Cost } \frac{2}{9} \text{ rods} = 40s.$$

$$\therefore \quad ,, \quad 1 \text{ rod} = \frac{40s. \times 9}{2} = £9.$$

To find the cartage per rod.

Wt. bricks in 1 rod $= 272 \times 120$ lbs.

$$\therefore \text{ Cost cartage} = \frac{272 \times 120}{2240} \times 10d. = 12s.\ 1\tfrac{3}{4}d.$$

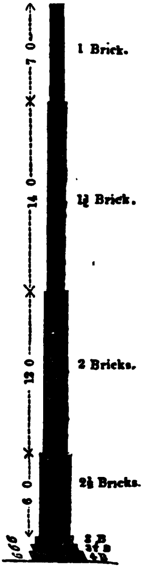
To find the cost of labour per rod.

Cost of $\tfrac{1}{2}$ rod $= 4s. \times 2 + 3s. \times 3 = 17s.$

$\therefore$ „ 1 rod $=$ £1 14*s.*

Collecting these items, we have,

	£	*s.*	*d.*
Cost bricks =	9	0	0
„ cartage =	0	12	$1\frac{3}{4}$
„ labour =	1	14	0
„ mortar =	0	5	0
	11	11	$1\frac{3}{4}$
Profit at 10 per cent =	1	3	$1\frac{1}{4}$
Charge per rod =	12	14	3



Example.—Required the cost of building the side wall of a house 40 ft. long, of which the annexed figure is a section, at £12 14*s.* per rod.

Here there are three footings, each $\frac{1}{2}$ ft. high. The first footing is 4 bricks in thickness, the second $3\frac{1}{2}$, and the third 3; hence we have the following areas in sq. ft. reduced to the standard thickness.

		sq. ft.
1st footing	$= 40 \times \frac{1}{2} \times \frac{8}{3} =$	53·3
2nd footing	$= 40 \times \frac{1}{2} \times \frac{7}{3} =$	46·6
3rd footing	$= 40 \times \frac{1}{2} \times \frac{6}{3} =$	40·0
Basement	$= 40 \times 6 \times \frac{5}{3} =$	400·0
1st floor	$= 40 \times 12 \times \frac{4}{3} =$	640·0
2nd floor	$= 40 \times 14 \times 1 =$	560·0
3rd floor	$= 40 \times 7 \times \frac{2}{3} =$	186·7
	Total content	1926·6

$\therefore$ No. rods $= 1926{\cdot}6 \div 272 = 7{\cdot}08308$.

$\therefore$ Cost $=$ £12 14*s*. $\times\ 7{\cdot}08308 =$ £89 19*s*. 1*d*.

In this calculation no deduction has been made for spaces occupied by windows or doors; however, it must be observed that all such apertures should be deducted, and a distinct charge made for window heads and sills, &c., or quoins, if any. Chimneys are measured as if they were solid, excepting the openings at the bottom.

CARPENTERS AND JOINERS' WORK.

11. Timber is sold to the carpenter in the form of logs, battens, deals, planks, and round or unsquared timber. When a log is of uniform breadth and thickness, its content is found by multiplying the length, breadth, and thickness together; thus if a log is 20 ft. long, 1·5 ft. broad, and 1·2 ft. thick, the content $= 20 \times 1{\cdot}5 \times 1{\cdot}2 = 36$ cubic feet.

12. When the breadth and thickness of a log are not uniform, it is customary to *multiply the mean breadth by the mean thickness, and this product by the length for the solidity.* For example, let the length $= 17\frac{1}{4}$ ft.; at the greater end, the breadth $= 3$ ft., and the thickness $= 1$ ft. 8 in.; and at the less end, the breadth $= 1$ ft. 6 in., and the thickness $= 10$ in. Here we have,

$$\text{Mean breadth} = \tfrac{1}{2}(3\text{ ft.} + 1\text{ ft. }6\text{ in.}) = 2\tfrac{1}{4}\text{ ft.}$$

$$\text{,, thickness} = \tfrac{1}{2}(1\text{ ft. }8\text{ in.} + 10\text{ in.}) = 1\tfrac{1}{4}\text{ ft.}$$

$$\therefore \text{Content} = 2\tfrac{1}{4} \times 1\tfrac{1}{4} \times 17\tfrac{1}{4} = 48{\cdot}515\text{ c. ft.}$$

The true content, measured as a frustum of a pyramid, (see Art. **22.** p. 170.), is about $\frac{1}{2}$ a cubic foot greater than this result; but it would seem that this deficiency in the measurement, in part compensates for the waste arising from the irregular shape of the log.

13. The content of unsquared timber is usually found, by *multiplying the square of $\frac{1}{4}$ the middle or mean girt, by the length.* Example.—The length of a tree is 32 ft., and its girt in the middle, after allowing for the bark, is 5 ft.; required the solid content.

$$\text{Here the square of } \tfrac{1}{4} \text{ mean girt} = (\tfrac{5}{4})^2 = \tfrac{25}{16},$$

$$\therefore \text{Content} = \tfrac{25}{16} \times 32 = 50 \text{ c. ft.}$$

14. Timber in logs is sold by the load of 50 cubic feet; and sometimes by the cubic foot. Battens are about 6 inches wide, and $2\frac{3}{4}$ inches thick; they are of various lengths. Deals are 9 inches wide and 3 inches thick; and planks are 11 inches wide. These are usually sold by the hundred (120), and sometimes by the running foot.

15. *To find the number of loads in* 1 *hundred* (120) 12 *ft.* -3 *inch deals* 9 *inches wide.*

$$\text{Content of deals} = 12 \times \tfrac{3}{12} \times \tfrac{9}{12} \times 120 = 270 \text{ c. ft.}$$

$$\therefore \text{No. loads} = 270 \div 50 = 5\tfrac{2}{5}.$$

16. A reduced or standard deal board is $1\frac{1}{2}$ in. thick, 11 in. wide, and 12 ft. long.

To reduce a given deal to the standard.

Let the given deal be 18 ft. long, 2 in. thick, and 9 in. wide.

$$\text{Content given deal} = 18 \times \tfrac{2}{12} \times \tfrac{9}{12} = \tfrac{9}{4} \text{ c. ft.}$$

$$\text{,, \quad standard deal} = 12 \times \frac{1\frac{1}{2}}{12} \times \frac{11}{12} = \frac{11}{8} \text{ c. ft.}$$

$$\therefore \text{Given deal reduced} = \tfrac{9}{4} \div \tfrac{11}{8} = 1{\cdot}636.$$

17. *Estimate of a cubic foot of timber in given scantlings.*

1. The timber cost £4 10*s.* per load, including cartage. It is in two logs, each 25 ft. long, and therefore 1 sq. ft. in the section; it is sawn into 20 scantlings by seven saw cuts, at the rate of 3*s.* per 100 sq. ft.* Required the charge for a cubic foot, allowing $\frac{1}{8}$ of an inch for each saw cut.

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* This charge is considerably too much, where the timber is cut by a *saw-mill.*

	£	s.	d.
Cost sawing $= \frac{1 \times 50 \times 7}{100} \times 3s. =$	0	10	6
Cost timber and cartage =	4	10	0
	5	0	6
Profit at 20 per cent. =	1	0	1
Total charge	6	0	7

$$\text{Waste by the saw cuts} = \frac{1 \times 50 \times 7}{8 \times 12} = 3{\cdot}64 \text{ c. ft.}$$

∴ Content of scantlings $= 50 - 3{\cdot}64 = 46{\cdot}35$ c. ft.

∴ Price of 46·35 c. ft. = £6 0*s.* 7*d.*

∴ „ „ 1 c. ft. = £6 0*s.* 7*d.* ÷ 46·35 = 2*s.* 7*d.*

A greater allowance should be made for the waste of timber, when the log is not uniform in its dimensions.

18. *Estimate of a sup. ft. of deal of a given thickness.*

1. 120 deals 12 ft. long, 3 in. thick, and 9 in. wide, cost £40. The deal boards are formed by 3 saw cuts of the 3 in. deal, each board being about $\frac{5}{8}$ in. thick, after allowing for the saw cuts. Required the charge per sup. ft. of the $\frac{5}{8}$ in. board, the rate of sawing being the same as the preceding article.

$$\text{Cost sawing} = 12 \times \tfrac{9}{12} \times 3 \times 120 \times \frac{3s.}{100} = 97{\cdot}2s.$$

No. sup. ft. in the boards $= 12 \times \frac{9}{12} \times 4 \times 120 = 4320$,

∴ Cost 4320 sup. ft. = £40 + 97·2*s.* = 897·2*s.*

∴ Cost 1 sup. ft. = 897·2*s.* ÷ 4320 = $2\frac{1}{2}$*d.*

In a similar manner the cost of any other thickness of deal may be determined.

19. *Estimates of joisting.*

1. The joists, not exceeding 9 in. deep, are trimmed to

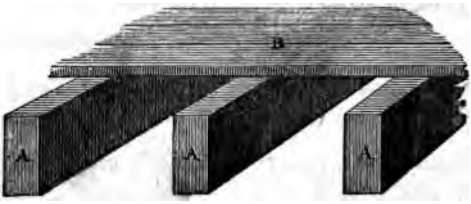
chimneys, &c.; the floor is single. Two workmen finished 660 sq. ft. of this joisting in $4\frac{1}{2}$ days. Required the number of days' work to 1 square.

$$\text{No. days' work to 660 sq. ft.} = 4\tfrac{1}{2} \times 2 = 9,$$

$$\therefore \quad ,, \quad ,, \quad \text{100 sq. ft.} = \frac{9 \times 100}{660} = 1{\cdot}36.$$

This result will form a constant factor, in calculating the cost of all such cases of joisting. Similar estimates may be made for other cases.

2. In a single floor there are 27 joists each 18 ft. long, 9 in. deep, and 3 in. thick. The floor is 40 ft. long; the price of the timber is 2*s.* 7*d.* per c. ft.; and the labour as given above. Required the cost when the workman receives 4*s.* per day.



$$\text{No. squares} = 40 \times 18 \div 100 = 7{\cdot}2,$$

$$\therefore \text{Cost labour} = 1{\cdot}36 \times 7{\cdot}2 \times 4s. = \pounds 1 \ 19s. \ 2d.$$

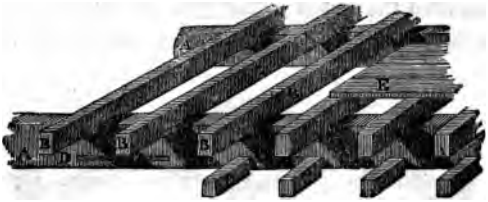
$$\text{No. c. ft. timber} = \frac{9 \times 3 \times 18 \times 27}{144} = 91\tfrac{1}{8},$$

$$\therefore \text{Cost timber} = 2s. \ 7d. \times 91\tfrac{1}{8} = \pounds 11 \ 15s. \ 4\tfrac{3}{4}d.$$

$$\therefore \text{Total cost} = \pounds 13 \ 14s. \ 6\tfrac{3}{4}d.$$

3. Required the cost of a double floor, the timber being at 2*s.* 6*d.* per c. ft., the constant of labour 3·58 days per square, and the wages 4*s.* per day. The dimensions are as follows: there are 4 *girders*, A, A, &c., supporting the floor,

each 32 ft. long, and 16 in. by 14 in. There are 24 *bridging joists*, being placed 16 in. middle and middle, crossing the girders and running from end to end of the floor, each 52 ft. long, and $7\frac{1}{2}$ in. by $2\frac{1}{2}$ in. There are 24 ceiling joists, C, C, &c., for receiving the laths of the ceiling, each 50 ft. long, and 5 in. by 2 in.



$$\text{Girders} = 32 \times 16 \times 14 \times 4 \div 144,$$
$$\text{Bridging joists} = 52 \times 7\tfrac{1}{2} \times 2\tfrac{1}{2} \times 24 \div 144,$$
$$\text{Ceiling joists} = 50 \times 5 \times 2 \times 24 \div 144,$$
$$\therefore \text{Total content} = (28672 + 23400 + 12000) \div 144 = 444{\cdot}944,$$
$$\therefore \text{Cost timber} = 2s.\ 6d. \times 444{\cdot}944 = £55\ 12s.\ 4\tfrac{1}{4}d.$$
$$\text{Cost labour} = \frac{32 \times 52}{100} \times 3{\cdot}58 \times 4s. = £11\ 18s.\ 3\tfrac{1}{4}d.$$
$$\therefore \text{Total cost} = £67\ 10s.\ 7\tfrac{1}{2}d.$$

20. *Estimates of roofing, &c.*

1. Three men framed and fixed in 15 days a king-post roof, containing 1290 sup. ft., (see next fig.). Required the number of days' work to 1 square.

No. days' work fixing 1290 ft. = 3 × 15 = 45,

" " 100 ft. = 45 × 100 ÷ 1290 = 3·48.

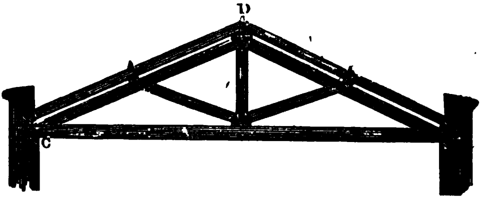
2. Required the cost of the labour in framing a king-post roof, 50 ft. long and 30 ft. in the girt; the wages of the workman being 3*s.* 6*d.* per day, and the constant of labour as just found.

No. squares in the roof = 50 × 30 ÷ 100 = 15,

$\therefore$ No. days' work $= 3{\cdot}48 \times 15 = 52{\cdot}2$,

$\therefore$ Cost $= 3s.\ 6d. \times 52{\cdot}2 = £9\ 2s.\ 8\frac{1}{4}d.$

3. Required the cost of a king-post roof having the following dimensions, allowing the wages to be 3*s.* 6*d.* per day, and the timber at 2*s.* 6*d.* per c. ft. The length of the *tie-beam* C C is 28 ft. and 10 in. by 5 in. The *king-post* I K $6\frac{1}{4}$ ft. and 5 in. by 5 in. The *principal rafter* C I $13\frac{1}{2}$ ft., and 5 in. by 4 in. The *struts* A K $6\frac{1}{4}$ ft. and 5 in. by 3 in. The *common rafters* D B $14\frac{1}{2}$ ft. and 4 in. by $2\frac{1}{2}$ in.; the number in the whole roof being 80. There are 5 *trusses* or couples C I C. The *purlines* A, running the whole length of the roof, 62 ft., and 8 in. by 6 in. The *ridge piece* D 62 ft. long, and 12 in. by $1\frac{1}{2}$ in. The *pole plate* B 62 ft. long, and 5 in. by $2\frac{1}{2}$ in. The two wall plates C and C, each 62 ft. long, and 2 in. by $2\frac{1}{2}$ in.



To find the c. ft. in each truss.

No. c. ft. in tie-beam $= 10 \times 5 \times 28 \div 144$,

" king-post $= 5 \times 5 \times 6\frac{1}{4} \div 144$,

" principal rafters $= 5 \times 4 \times 13\frac{1}{2} \times 2 \div 144$,

" struts $= 5 \times 3 \times 6\frac{1}{4} \times 2 \div 144$.

Adding these together we find 15·859 c. ft. in each truss

$\therefore$ No. c. ft. in the 5 trusses $= 15{\cdot}859 \times 5 = 79{\cdot}29$.

To find the c. ft. in the other parts of the roof.

Common rafters $= 4 \times 2\frac{1}{2} \times 14\frac{1}{2} \times 80 \div 144$,

Purlines $= 8 \times 6 \times 62 \times 2 \div 144$,

Ridge piece $= 12 \times 1\frac{1}{2} \times 62 \div 144,$
Pole plates $= 5 \times 2\frac{1}{2} \times 62 \times 2 \div 144,$
Wall plates $= 2 \times 2\frac{1}{2} \times 62 \times 2 \div 144.$

Adding these together we find 144·707 c. ft. for the sum. Which added to the content of the trusses gives 223·997 c. ft. for the timber in the whole roof.

∴ Cost timber $= 2s.\ 6d. \times 223{\cdot}997 =$ £27 19*s.* 11¾*d.*

To find the cost of the labour.

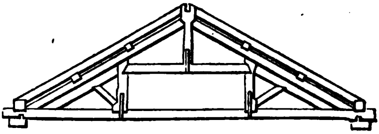
No. squares $= 62 \times 14\frac{1}{2} \times 2 \div 100 = 17{\cdot}98,$
No. days' work $= 3{\cdot}48 \times 17{\cdot}98,$
∴ Cost labour $= 3{\cdot}48 \times 17{\cdot}98 \times 3s.\ 6d. =$ £10 18*s.* 11¾*d.*

Adding the cost of the timber and labour together, we find the whole cost of the roof to be £38 18*s.* 11½*d.*

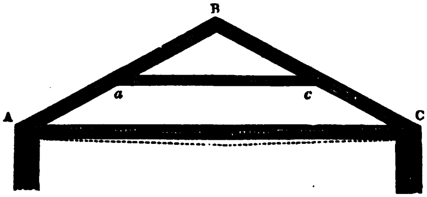
In this calculation the cost of laths, eave-boards, &c., nails, straps, and bolts, has not been included.

4. Required the cost per square in the last example.
Ans. £2 3*s.* 3¾*d.*

5. Required the cost in fixing a *queen-post* roof, framed as in the annexed figure, the length being 60 ft., the girt 45, the labour in one square 4·55 days, and the wages of a man 3*s.* 4*d.* per day. *Ans.* £20 9*s.* 6*d.*



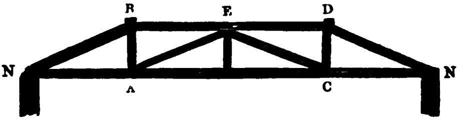
6. Required the cost of the labour in fixing a roof 36 ft. long, and 9 ft. in each rafter; where the truss has simply the principal rafters A B and B C, the tie-beam A C, the *collar* beam *a c*, and



purlins; allowing that a man takes 2·323 days to complete a square, and that he receives 5*s.* per day. *Ans. £*3 15*s.* 3*d.*

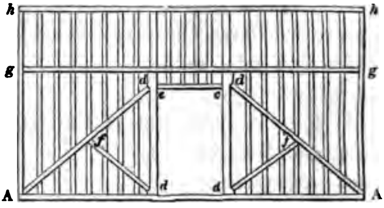
7. Required the cost of the labour in a square of common shed roofing one story high, allowing ·968 days as the labour per square, and 4*s.* 6*d.* per day for the workman? *Ans.* 4*s.* 4¼*d.*

8. Required the number of c. ft. in a queen-post truss, from the following dimensions. Tie-beam N N 43 ft. and 12 in. by 5 in. The two *queen-posts* A B and C D, allowing for tenons, each 5½ ft. and 5 in. by 5 in. The *collar* B D 20 ft. and 7 in. by 5 in. The struts A E and C E, each 10 ft. and 5 in. by 2½ in. The principal rafters N D each 12 ft. and 5 in. by 5 in. The king-post E 4½ ft. and 5 in. by 5 in. The annexed figure is constructed on a scale of a half inch to 10 ft. *Ans.* 31·36 *c. ft.*



9. The length of the angles of a hipped roof is 40 ft., required the extra charge for framing the roof, allowing ·145 days per running foot, the wages being 5*s.* per day. *Ans. £*1 9*s.*

10. Required the cost of the labour in fitting the partition A A *h h*, 24 ft. long, and 13 ft. high; having the *head h h*, the *sill* A A, the door-posts *d c*, the *inter-tie g g*, the *braces* A *d* and A *d*, the *struts f d* and *f d*; allowing 1·743 days for a square, and 4*s.* 6*d.* per day wages. *Ans. £*2 4*s.* 5½*d.*



21. *Estimates of framed work.*

1 A man was found to frame 7 doors in 13¼ days. Each

door was 6 ft. 9 in. by 3 ft. 6 in. and $1\frac{1}{4}$ in. thick, and had six plain panels. Required the constant of labour per sq. ft.

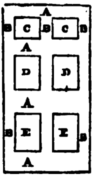
$$\text{Sup. ft. in 7 doors} = 6\tfrac{3}{4} \times 3\tfrac{1}{2} \times 7 = 165{\cdot}375,$$

$$\therefore \text{ No. days for } 165{\cdot}375 \text{ sq. ft.} = 13{\cdot}25,$$

$$\therefore \quad ,, \qquad ,, \quad 1 \text{ sq. ft.} = 13{\cdot}25 \div 165{\cdot}375 = {\cdot}08.$$

2. Required the cost of the labour per sq. ft., when the joiner receives 5*s.* per day. *Ans.* $4\frac{3}{4}$*d.*

3. In a door 6 ft. 9 in. by 3 ft. 6 in. having six panels; the *styles* B, B, B are cut out of stuff 20 ft. long, and 4 in. by $1\frac{1}{2}$ in.; the two top *rails* A, A 7 ft. long, and 5 in. by $1\frac{1}{2}$ in.; the two bottom rails 7 ft. long, and 9 in. by $1\frac{1}{2}$; the panels C, C, D, D, E, E $9\frac{1}{2}$ ft. long, and 9 in. by $\frac{1}{2}$ in. Required the total cost of the door, the labour being ·08 days per sq. ft., the wages 5*s.* per day, the $1\frac{1}{2}$ in. stuff at 8*d.* per sup. ft., and the $\frac{1}{2}$ in. at $3\frac{1}{2}$*d.*



To find the cost of the timber.

No. sup. ft. in the $1\frac{1}{2}$ in. stuff $= 20 \times \frac{4}{12} + 7 \times \frac{5}{12} + 7 \times \frac{9}{12}$ $= \frac{1}{12}(80 + 35 + 63) = 14\frac{5}{6}$.

$$\therefore \text{ Cost } 1\tfrac{1}{2} \text{ in. stuff} = 14\tfrac{5}{6} \times 8d. = 9s.\ 10\tfrac{1}{2}d.$$

No. sup. ft. in the $\frac{1}{2}$ in. stuff $= 9\frac{1}{2} \times \frac{3}{4} = 7\frac{1}{8}$,

$$\therefore \text{ Cost } \tfrac{1}{2} \text{ in. stuff} = 3\tfrac{1}{2}d. \times 7\tfrac{1}{8} = 2s.\ 1d.$$

$$\therefore \text{ Total cost of timber} = 11s.\ 11\tfrac{1}{2}d.$$

To find the cost of the labour.

$$\text{No. sup. ft. in the door} = 6\tfrac{3}{4} \times 3\tfrac{1}{2} = \tfrac{189}{8},$$

$$\therefore \text{ Cost labour} = {\cdot}08 \times \tfrac{189}{8} \times 5s. = 9s.\ 5\tfrac{1}{4}d.$$

Hence we find the total cost to be £1 1*s.* $4\frac{3}{4}$*d.*

4. Required the cost per sup. ft. in the last example.

Ans. $10\frac{3}{4}$*d.*

5. Required the cost of the labour in a door 7 ft. by 3 ft. 6 in., having 4 panels, quirked ovolo and bead on both sides, allowing ·15 days for 1 sup. ft., and 4*s.* per day for the workman. *Ans.* 14*s.* 8¼*d.*

22. *Estimates of flooring.*

1. A joiner prepared in 1 day 12 flooring boards, each 12 ft. long, 9 in. wide, and 1¼ in. thick. Required the constant of labour per sup. ft.

$$\text{No. sup. ft. in the boards} = 12 \times \tfrac{9}{12} \times 12 = 108,$$

$$\therefore \text{ No. days for 108 sup. ft.} = 1,$$

$$\therefore \quad ,, \quad ,, \quad \text{1 sup. ft.} = 1 \div 108 = \cdot00926.$$

2. Required the cost of flooring a room 15 ft. square with 1¼ in. battens 6 in. wide when jointed. The rough batten costs 4*d.* per running foot, and each will cut into two flooring boards. The sawing costs ¼*d.* per sup. ft. A workman will prepare in 1 day for flooring 22 boards each 10 ft. long, and will lay down the given floor in 2½ days; his wages being 3*s.* 6*d.* per day, and the cost of nails 2*s.*

$$\text{No. sup. ft. in the room} = 15 \times 15 = 225,$$

$$\text{Cost timber for 1 sup. ft.} = 4d.$$

$$,, \quad \text{225 sup. ft.} = 4d. \times 225 = £3\ 15s.$$

$$\text{Cost sawing} = \tfrac{1}{4}d. \times \tfrac{1}{2} \text{ of } 225 = 2s.\ 4d.$$

$$\text{Cost preparing 110 sup. ft.} = 3s.\ 6d.$$

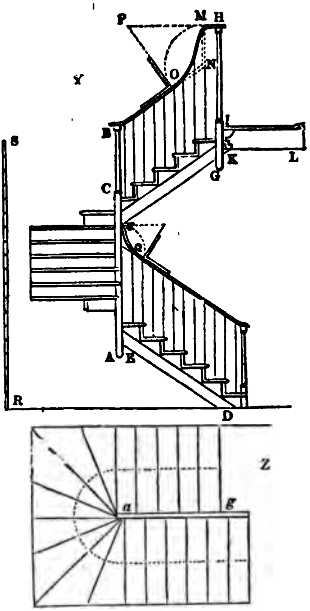
$$\therefore \quad ,, \quad \text{225 sup. ft.} = \frac{3s.\ 6d. \times 225}{110} = 7s.\ 1\tfrac{3}{4}d.$$

$$\text{Cost laying down} = 3s.\ 6d. \times 2\tfrac{1}{2} = 8s.\ 9d.$$

Adding these various items together with 2*s.* for nails, we find the total cost to be £4 15*s.* 2¾*d.*

3. Required the cost per square in the last example. *Ans.* £2 2*s.* 3½*d.*

23. *Estimate of the labour in a dog-legged staircase,* of



which the annexed cut is the plan and elevation, on a scale of an eighth of an inch to a foot. In this staircase the *flyers* extend from D to A, and from C to I; the *winders* or oblique steps from A to C. The *tread* is the upper part of the step, and the *riser* the front part. DE and FG are the *string boards*, framed into the *newel* BA; Q and BOM are the *hand-rails*. The risers, treads, and string-boards are usually measured by the superficial foot. The price depends upon the form of the stairs, whether it be dog-legged, or geometrical having a well-hole. The hand-rail and newel are measured by the lineal foot; and brackets, balusters, rails, &c., are valued by the piece.

1. Required the cost of fixing the flyers, winders, and string-boards, of the staircase represented in the annexed figure; the labour in the flyers with string-boards being ·103 days per sup. ft., and in the winders with string-boards ·135 days per sup. ft.; taking the man's wages at 3*s.* per day.

There are 86 sup. ft. in the flyers with string-boards.

∴ No. days fixing the flyers = ·103 × 86 = 8·858.

There are 60 sup. ft. in the winders with string-boards.

∴ No. days fixing the winders = ·135 × 60 = 8·1,

∴ Total no. days = 8·858 + 8·1 = 16·958,

∴ Cost labour = 3*s.* × 16·958 = £2 10s. 10½d.

2. If the hand-rail is 13½ ft. long, what will be the cost of the labour, allowing ·111 days per running ft.?

Ans. 4*s.* 5¾*d.*

The various illustrations of the method of estimation, which have been given, will show the use of the constants of labour; an extensive list of them, applicable to almost every kind of carpenter and joiner's work, will be found in "Gwilt's Encyclopædia of Architecture." Much valuable information on building may also be obtained from "The Mechanic's Magazine," and "The Builder."

To determine from a London "price book" the value of any kind of work in the country.

24. In the following calculations, we shall suppose the London wages to be 5*s.*, and the country wages 4*s.* per day; the rate of profit in both cases to be 10 per cent; and the cost of the material to be the same in both cases, excepting where mention is made to the contrary. It is also taken for granted, that the workman is able to measure and estimate the value of the material, which may be used in a given dimensions of the particular kind of work.

1. If the price of labour per sup. ft. in framing window-shutters be 1*s.* 6*d.*; required the charge in the country.

$$\text{Price at } 5s.\text{ wages} = 1s.\ 6d.$$

$$\therefore \quad ,, \quad 1s.\text{ wages} = \frac{1s.\ 6d.}{5}$$

$$\therefore \quad ,, \quad 4s.\text{ wages} = \frac{1s.\ 6d. \times 4}{5} = 1s.\ 2\tfrac{1}{4}d.$$

2. If the price of wainscoting per sup. ft. be 2*s.*, and the cost price of the timber in 15 sup. ft. 5*s.*; required the country charge per sup. ft.

$$\text{Cost timber per sup. ft.} = 5s. \div 15 = 4d.$$

$$\text{Charge} \quad ,, \quad = 4d. + \tfrac{1}{10}\text{ of }4d. = 4\tfrac{1}{4}d.$$

$$\therefore \textit{Charge for labour} \quad ,, \quad = 2s. - 4\tfrac{1}{4}d. = 1s.\ 7\tfrac{3}{4}d.$$

But this is the charge when the wages are 5*s.* per day.

∴ Charge labour at 4*s.* per day = ⅘ of 1*s.* 7¾*d.*

This result being added to the charge for the material, will give 1*s.* 8*d.* as the charge per sup. ft. in the country.

3. If the price of 1¼ in. batten deal flooring be £3 per square, and the cost price of a sup. ft. of the timber 4*d.*; what will be the charge per square in the country?

Cost timber per sq. = 4*d.* × 100 = 400*d.*
∴ Charge „ = 400*d.* + ⅒ of 400*d.* = £1 16*s.* 8*d.*
∴ Charge labour „ = £3 − £1 16*s.* 8*d.* = £1 3*s.* 4*d.*

But this is the charge of labour per square, when the workman receives 5*s.* per day.

∴ Charge labour at 4*s.* per day = ⅘ of £1 3*s.* 4*d.*

Adding the charge for the timber to this result, we find the charge per square for the country to be £2 15*s.* 4*d.*

4. If the price for a staircase be £8, and the cost price of the material £2; what will be the charge in the country?

Charge for material = £2 + ⅒ of £2 = £2 4*s.*
∴ Charge for labour = £8 − £2 4*s.* = £5 16*s.*
∴ Country charge for labour = ⅘ of £5 16*s.* = £4 12*s.* 9½*d.*

Adding the charge for material, we find the country charge for labour and material to be £6 16*s.* 9½*d.*

5. If the price for a square of roofing be 56*s.*, and the cost price of the material 40*s.* per square, what will be the charge per square in the country? *Ans.* £2 13*s.* 7*d.*

6. Required the same as in the last example, taking the cost price of the material in town at 35*s.*, when it is 45*s.* in the country.

Here we find, after the manner of the preceding examples, the charge of labour in the country to be 14*s.* We then find the charge for the material = 45*s.* + ⅒ of 45*s.* = 49·5*s.* = £2 9*s.* 6*d.*; and the total charge = £3 3*s.* 6*d.*

7. Required the same as in Ex. 5., supposing the profit on the material to be 20 per cent., and on the labour 10 per cent? *Ans.* £2 14*s.* $4\frac{3}{4}$*d.*

8. If the price of a sup. ft. of sashes and frames be 2*s.*, and the cost price of the material for 24 sup. ft. 8*s.*; what will be the price per sup. ft. in the country, allowing the profit there to be 5 per cent., while it is 10 per cent. in town?

Cost price $= 2s. \times 100 \div 110 = 1s.\ 9\frac{3}{4}d.$
Cost material per sup. ft. $= 8s. \div 24 = 4d.$
$\therefore$ Cost labour $= 1s.\ 9\frac{3}{4}d. - 4d. = 1s.\ 5\frac{3}{4}d.$
$\therefore$ Cost labour in the country $= \frac{4}{5}$ of $1s.\ 5\frac{3}{4}d. = 1s.\ 2\frac{1}{4}d.$

$\therefore$ Cost labour and mat. $= 1s.\ 2\frac{1}{4}d. + 4d. = 1s.\ 6\frac{1}{4}d.$
Profit at 5 per cent. $= 0s.\ 0\frac{3}{4}d.$
$\therefore$ Charge required $= 1s.\ 7d.$

PLASTERERS' WORK.

25. Plasterers' work is usually estimated by the square yard. In forming an estimate of the value of a sq. yd. of plastering, the different items of expense should be calculated after the manner explained in the preceding articles.

EXAMPLE. — Required the charge for plastering a room 18 ft. long, 15 ft. broad, and 9 ft. high. The walls to have two coats of plaster, at 5*d.* per sq. yd.; and the ceiling to be lath and plaster, at 11*d.* per sq. yd.; making a deduction in the walls for a door 7 ft. by 4 ft., and for a fireplace 4 ft. 6 in. by 4 ft. *Ans.* £2 12*s.* $10\frac{1}{4}$*d.*

SLATERS' WORK.

26. Tiling is measured by a square of 100 superficial feet. In measuring a slate, or plain tile, roof, 6 inches are allowed for the double slates or plain tiles, which are put at the *eaves.*

Estimate of a Square of Slating.

Let us assume, for the sake of illustration, the following data:— The cost and cartage of the slates to be £3 per thousand, and each slate, having a 5-inch gauge, to cover on an average 24 sup. in. A man at 3*s.* per day will dress, lay on, and point $\frac{3}{4}$ of a square in 1 day. Required the charge for a square of slating.

To find the cost of the slates per square.

No. slates to cover 100 ft. $= 100 \times 144 \div 24 = 600$,

Cost of 1000 slates = £3

∴ Cost of 600 slates = £3 × 600 ÷ 1000 = £1 16*s.*

To find the cost of the labour per square.

Cost of $\frac{3}{4}$ of a square = 3*s.*

" 1 square = 3*s.* × 4 ÷ 3 = 4*s.*

Collecting these items, we have,

	£	*s.*	*d.*
Material and cartage	1	16	0
Labour - -	0	4	0
Nails and mortar -	0	2	0
	2	2	0
Profit at 10 per cent.	0	4	2½
Charge per square -	2	6	2½

Example. — Required the charge for slating a roof 20 ft. long, and 9 ft. from the ridge to the eaves, allowing in addition 6 in. on each side for the double slates at the eaves, at £2 6*s.* per square. *Ans.* £8 14*s.* 9½*d.*

PAINTERS AND GLAZIERS' WORK.

27. Painters usually estimate their work by the square yard, and glaziers by the superficial foot. An analysis of the elements of costs, in these different works, may be made, after the manner pursued in the foregoing pages,

Ex. 1. A door measures 7 ft. 3 in. by 3 ft. 9 in., what did it cost painting, on both sides, with three coats, at 9*d.* per sq. yd.? *Ans.* 4*s.* 6½*d.*

2. The compass of a wainscotted room is 80 ft. and its height 10 ft. 6 in.; what did it cost painting, with one coat, at 5*d.* per sq. yd.; deducting a fireplace 5 ft. by 6 ft., and a window 6 ft. by 3 ft. 6 in.? *Ans.* £1 16*s.* 6¼*d.*

3. Required the charge for glazing a window 4 ft. 6 in. by 3 ft., at 1*s.* 6*d.* per square foot. *Ans.* £1 0*s.* 3*d.*

4. A pane of plate glass measures 3 ft. by 1 ft. 4 in.; what did 4 such panes cost, at 7*s.* 3*d.* per square foot? *Ans.* £5 16*s.*

5. A house has two tiers of windows, and four in each tier; the height of glass in the first is 6 ft., and in second 5 ft., and the breadth of the glass in each window is 3 ft. 3 in.; what did the whole cost glazing at 1*s.* 8*d.* per sq. ft.? *Ans.* £11 18*s.* 4*d.*

6. A window has 16 panes, each measuring 14 in. by 9 in.; required the charge for glazing, at 1*s.* 4*d.* per sq. ft. *Ans.* 18*s.* 8*d.*

PLUMBERS, FOUNDERS, AND SMITHS' WORK.

28. Plumbers' work is charged by the cwt. or lb., to which is added the labour of laying the lead. Water pipes are charged by the foot, according to their diameter. The weight of a pipe depends upon its bore; leaden pipes of ¾ inch bore weigh 10 lbs., of 1 inch bore 12 lbs., of 1¼ inch bore 16 lbs., of 1½ inch bore 18 lbs., of 1¾ inch bore 21 lbs., and of 2 inch bore 24 lbs. per running yard. Sheet lead varies from 5 to 12 lbs. per superficial foot, and is used for covering flats in roofs, gutters, &c. The weight of a cubic foot of lead being nearly 708 lbs., the weight of a superficial foot of sheet lead of any given thickness may be readily found. Thus, for example, let it be required to find the weight of a sup. ft. of sheet lead $\frac{1}{4}$ inch thick.

Wt. sup. ft. 12 in. thick = 708 lbs.
∴ „ „ 1 „ = 708 ÷ 12 = 59 lbs.
∴ „ „ $\frac{1}{4}$ „ = $\frac{1}{4}$ of 59 = $14\frac{3}{4}$ lbs.

Conversely, when the weight of a sup. ft. is given, we can readily find the thickness. Thus, for example, let it be required to find the thickness when the weight of a sup. ft. is 10 lbs.

Thickness when its wt. is 708 lbs. = 12 in.
∴ „ „ „ 1 lb. = 12 ÷ 708 = $\frac{1}{59}$
∴ „ „ „ 10 lbs. = $\frac{10}{59}$ = $\frac{1}{6}$ in. nearly.

Founders and smiths' work is generally charged by the cwt. or lb. Locks, latchets, &c. are charged so much for each, according to the size and quality; hinges by the pair; and nails by the hundred.

Ex. 1. Required the charge for lining a water cistern 3 ft. long, 2 ft. deep, and 2 ft. 6 in. broad, with sheet lead of 10 lbs. per sup. ft., at £2 per cwt. *Ans.* £5 5*s.* $4\frac{1}{4}$*d.*

2. Required the charge for laying a gutter 40 ft. long, 1 ft. 6 in. broad, with sheet lead 9 lbs. to the sup. ft.; the flashing being 46 ft. long and 6 in. broad, with lead 6 lbs. to the sup. ft.; at £1 18*s.* per cwt. *Ans.* £11 10*s.* $0\frac{1}{4}$*d.*

3. Required the charge per running foot for a leaden pipe of $1\frac{1}{2}$ inch bore, at 4*d.* per lb. *Ans.* 2*s.*

4. What will be the charge for iron straps and bolts, weighing 2 cwts. 16 lbs. at $3\frac{1}{2}$*d.* per lb.? *Ans.* £3 10*s.*

LAND-SURVEYING.

1. LAND-SURVEYING is the method of measuring and computing the area of any portion of ground, as a field or an estate. The most useful instruments used in surveying are the chain, the offset staff, the cross staff, and the theodolite.

2. The chain is composed of 100 links, and at every tenth link is a piece of brass, to assist the surveyor in reckoning the number of links that may be measured. (See p. 140.) In measuring with the chain, ten iron pins or *arrows* are used for sticking into the ground at every chain's length. *Offset staves* are rods ten links long; they are divided into links, for measuring short distances, such as offsets. *Piquet staves* or *station staves* are used for marks to be placed at the stations or remarkable places, such as the corners of fields, &c. *A chain line* is the straight line which joins any two extreme stations. *Offsets* are the perpendicular distances measured from the chain-line to the boundary of a field. *The cross staff* is chiefly used in raising short perpendiculars to the chain line, such as *offsets*. (See Geo. p. 8.)

3. The measurements taken in a survey are usually noted down in a tabular form, called the *field-book*. In order to indicate the direction of the measurements, the field-book begins at the bottom of the page. The page of the field-book is divided into three columns. The middle column contains the lengths measured on the chain lines, and the columns to the right and left hand contain respectively the *right* and *left offsets*, &c., taken at the corresponding points of the chain-line. The station from which the measurements commence is called the first station; the next arrived at, the second; and so on. The stations are usually indicated *in their order* by the letters A, B, C, &c., or by the symbols

$\odot_1$, $\odot_2$, $\odot_3$, &c. Sometimes also the small letters *a*, *b*, *c*, &c., are used to denote the order of the minor stations. The station staves, as they are put into the ground, should be marked with the letter or number which is entered in the field-book.

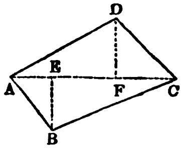
TO SURVEY WITH THE CHAIN.

4. Divide the field, or piece of ground, into triangles, or into triangles and four-sided figures; and place station staves at every corner. The principal triangles should occupy the main body of the field, and then the remaining portions will have the form of small triangles and trapezoids, which may be measured by means of offsets. When the principal triangles are large, and great accuracy is required, the three sides of each should be measured on the ground. But in the case of small triangles the areas may be found sufficiently exact by measuring a base and perpendicular.

In order to measure the distance between two stations, the assistant, or "leader," takes nine arrows in his left hand, and one end of the chain and the other arrow in his right; then advancing in a straight line towards the station-staff, at the end of the chain, he puts down the arrow which he held in his chain hand. The "follower," or surveyor, takes up this arrow, when he comes to it, in his chain hand, at the same time the leader puts down another arrow. When the leader has put down his last or tenth arrow, he advances a chain further, and setting his foot upon the end of the chain, he calls out "change;" the follower then walks up to the leader and delivers the ten arrows to him; one of them is then put down at the end of the chain. At every call of "change" the surveyor enters 1000 in his field-book. Before any arrow is put into the ground, the follower must see that it is in a line with the station-staff, observing to wave his hand to the right or to the left, as the case may require should the arrow not be in the proper line.

Sometimes it is requisite that *the offsets, to the brow of the ditch, as well as the offsets to the fence, should be entered in the field-book;* in such cases the offset is taken to the roots of the quickwood, and 5 or 6 links, according to the custom of the place, are added or subtracted for the breadth of the ditch. If a ditch is in the field which is being surveyed, the hedge and ditch usually belong to the adjoining field.

Let it be required to find the dimensions of the field represented by the figure ABCD.



Here dividing the field into two triangles by the diagonal AC, the surveyor measures along this diagonal until he arrives at the point E, from which he supposes a perpendicular, EB, will cut the corner B; he plants his cross at E and turns its index until the station A or C can be seen through one of the grooves; he then applies his eye to the other groove, and if he sees the corner B he is in the right place for measuring the perpendicular EB; but if the corner B is not in a line with the groove, he must move the cross backward or forward on the chain-line AC, as the case may require. The distance AE must be entered in the middle column of the field-book, and the length of the offset EB in the right hand column of offsets. In like manner the distance AF must be entered in the middle column, and the length of the offset FD in the left hand column of offsets; and so on.

Let AE = 80, EB = 110, AF = 220, FD = 120, and AC = 340 links; then the field-book will have the following form:

Left Offsets.	Chain Line.	Right Offsets.
0	340 to C	0
120	220	
	80	110
0	From A.	0

To draw a plan of the ground. From a scale of about 2 inches to 100 links, take off AE = 80, and mark off the perpendicular EB = 110; take AF = 220, and the perpendicular FD = 120; take AC = 340; join AB, BC, CD, and DA; then ABCD will be the plan of the field.

The scale of equal parts should be divided into tenths, &c. A feather-edged scale, made of ivory or fine millboard paper, is the best instrument for laying down dimensions, for distances may be pricked off by it without using the compasses.

Hence we find, by Art. 4., p. 145., the area

$$= \tfrac{1}{2} \text{ of } 340 \times (110 + 120) = 39100 \text{ sq. links} = 1 \text{ r. } 22{\cdot}56 \text{ p.}$$

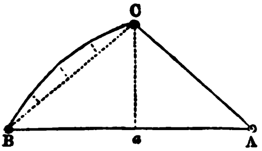
EXAMPLES.

1. Required the form of the field-book of Ex. 4. p. 147.

0	1110 to B.	0
595	745	
352	110	
0	From A.	0

2. Required the form of the field-book of Ex. 3. p. 145.
3. Draw a plan and find the area of a triangular field from the following field notes.

	330 to A.	
From	C on R.	
0	350 to C.	0
36	250	
40	170	
20	80	
From	B on R.	
	500 to B.	
Proof	250 to *a*.	line *a*C = 232.
From	A go W.	



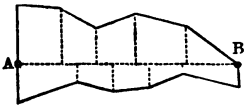
In this field-book there are three chain-lines. When a chain line is completed, a line is drawn through the middle column; thus the measurement from A to B forms a chain-line. When the surveyor arrives at B, he proceeds to C, which lies to the right hand of the direction in which he had been going; this is indicated in the field-book by writing "From B on R." The "proof line" ac is taken to check the work.

Construction. With a scale of equal parts, taking about 2 inches to 100 links, construct the triangle ABC, making AB = 500, BC = 350, and AC = 330; then draw the offsets upon BC. To verify the accuracy of the survey, take Aa = 250, and join Ca; then, if this "proof line" is found to measure 232, it will show that the dimensions of the field have been correctly taken. To facilitate the calculation, measure off the perpendicular let fall from C on AB (= 230).

Calculation. 2 Area ABC = 500 × 230 = 115000
„ offsets on BC = 16680
2|131680
Area = 2 r. 25·344 p. = 65840

4. Draw a plan and find the area of an irregular piece of ground from the following field-book:

0	1310 to B.	130
230	1000	
	980	50
	780	130
310	700	
	550	150
220	460	
	330	94
330	260	
360	0	230
	From A.	



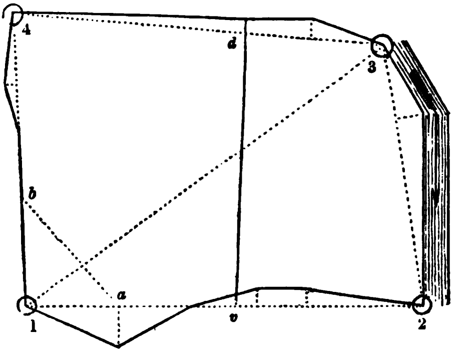
Area=4 *a.* 3 *r.* 16·24 *p.*

5. Draw a plan and find the area of a four-sided field from the following field notes:

	370 to A.
110	220
	From C on R.
	280 to C.
	From B on R.
	470 to B.
	From A.

Area=2 *r.* 35 *p.*

6. Draw a plan and find the area of two fields from the following notes. *Ans. 5a. 0r. 31p.*

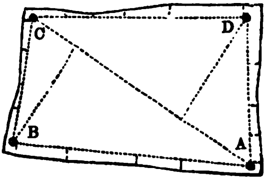


Left Offsets.	Chain Line.	Right Offsets.		
	950 to $\odot_3$.			
s ——	570	—— s		
Proof	145	line $ab = 307$.		
	From $\odot_1$.			
	620 to $\odot_1$.	0	38	
	400 to *b*.	0		
	260	0	40	to road.
	150	20	35	
	From $\odot_4$ on L.			
0	810 to $\odot_4$.	0		
s ——	300	$\overset{\frown}{24}$ —— s		
	160	40		
	From $\odot_3$ on L.			
	560 to $\odot_3$.	0		
	400	60	70	to river.
	From $\odot_2$ on L.			
0	860 to $\odot_2$.	0		
40	600			
40	500			
In line with fence	480			
Cross fence ⟍	350	to outside.		
	200 to *a*.	80 >		
From	$\odot_1$ going	East.		

In this field-book, the five chain-lines form the sides of two large triangles $\odot_1 \odot_2 \odot_3$ and $\odot_1 \odot_3 \odot_4$. At the first offset a mark is put to show the form of the fence. At 480 on the chain-line, the entry "In line with fence" indicates that a fence lying to the left is in a line with that point. When any fence is crossed, as at *d*, in $\odot_3 \odot_4$, a line is drawn on both sides of the corresponding distance in the chain-line; when the fence is straight from end to end, the lines are marked *s*, as in the field-book opposite to the distance 300. The position of the points *v* and *d* being ascertained on their respective chain-lines, the straight fence *d v* may be drawn upon the plan. When an offset is not taken perpendicular to the chain-line, but in the direction of a fence, the mark $\frown$ is placed over it, as $\overset{\frown}{24}$ at 300 in the chain-line $\odot_3 \odot_4$. The proof-line in this example is *ab;* its length and the point where it cuts the diagonal $\odot_1 \odot_3$ is entered in this last chain-line.

Construction. Here we have given all the sides in the triangles $\odot_1 \odot_2 \odot_3$ and $\odot_3 \odot_1 \odot_4$, viz., $\odot_1 \odot_2 = 860$, $\odot_2 \odot_3 = 560$, $\odot_3 \odot_4 = 810$, $\odot_4 \odot_1 = 620$, and $\odot_1 \odot_3 = 950$ links. Having constructed these triangles on a scale of about 2 inches to 100 links, the perpendiculars upon the diagonal $\odot_1 \odot_3$ will be found to be 501 and 524 links; hence the area of the trapezium $\odot_1 \odot_2 \odot_3 \odot_4$ will be readily found. The check-lines must now be drawn: take $\odot_1 a = 200$, and $\odot_4 b = 400$; join *ab;* then if this line, on the plan, measures 307, it will show that the dimensions in the field-book have been accurately measured. Take $\odot_1 v = 480$, $\odot_3 d = 300$, join *vd*, and it will give the interior fence. The offsets from the first four chain-lines may now be drawn.

7. Draw a plan and find the area of a field from the following field-book.



	1000 to C.
	From A on R.
30	630 to A.
40	500
30	400
40	0
	From D on R.
40	950 to D.
50	600
30	400
20	150
30	0
	From C on R.

30	550 to C.
10	400
20	160
20	0
	From B on R.
20	1050 to B.
60	800
40	500
50	300
20	0
From	A go West.

Here by constructing the figure on a scale of about 1 inch to 100 links, we find the perpendiculars on the diagonal to be 540 and 580 links. All the offsets will form trapezoids, the areas of which may be calculated by Art. 5. p. 146., excepting the spaces in the four corners, which will be trapeziums, and their areas should therefore be calculated by the dimensions taken from the plan. Some writers on surveying erroneously calculate these areas, by taking the product of the two offsets given in the field-book.

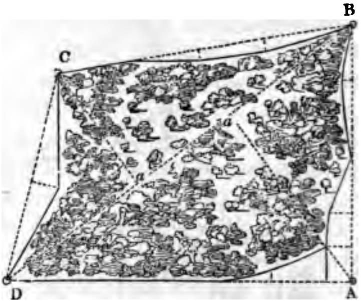
2 Area trapezium ABCD = 1120000
„ corners = 7300
„ offsets = 220200

2|1347500

Area = 6a. 2r. 38p. = 6·73750 a.

8. Draw a plan and find the area of a wood from the following field-book. *Ans.* 13*a.* 0*r.* 32*p.*

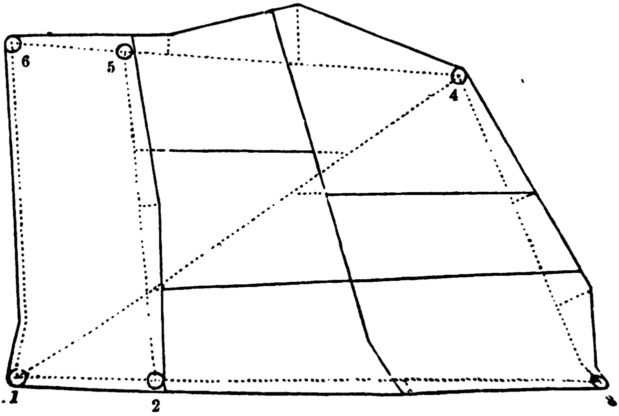
The station A is taken so that AB and AD are at right angles to each other. In this example, *insets* to the fence are measured from the chain-lines, AB, BC, CD, and DA.



	1560 to A.
150	1450
40	1200
0	890
0	0
	From D on L.
0	950 to D.
125	510
0	0
	From C on L.

0	1350 to C.
0	1060
40	700
50	390
0	0
	From B on L.
0	1150 to B.
50	905
0	560
100	300
110	175
	From A.

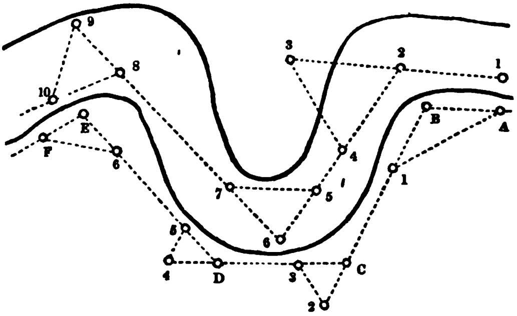
9. Draw a plan and find the area of an estate from the following field-book. *Ans.* 13*a.* 2*r.* 8*p.*



	Check Line.	
	875 to $\odot_5$.	
To corner	600	40 —— *s* fence.
	460	50
Line with	230	. . . —— *s* fence
From	$\odot_2$ to $\odot_5$.	
	1410 to $\odot_4$.	
Line with fence	1030	
s ——	940	—— *s*.
	850	. . . line fence.
Cross	420	fence.
From	$\odot_1$ on L.	
	890 to $\odot_1$.	20
	0	20
From	$\odot_6$ on L.	
	1200 to $\odot_6$.	25
	900 to $\odot_5$.	
Cross	880	fence.
	790	60
s ——	470	—— *s*.
	430	150
	0	20
From	$\odot_4$ on L.	
	900 to $\odot_4$.	10
s ——	560	—— *s*.
To end of	530	70 fence.
s ——	320	—— *s*.
	220	100
	0	20
From	$\odot_3$ on L.	
	1500 to $\odot_3$.	22
	980	30
s ——	930	—— *s*.
Cross	360	fence.
	300 to $\odot_2$.	25
	0	20
From	$\odot_1$ go East.	

When the chain-lines in a survey are *closed*, that is, when the measurements are sufficient to enable us to construct the lines, a double line is drawn through the middle column of the field-book.

10. Draw a plan of a road, on a scale of 2 inches to a chain, from the following field-notes.



Here we begin the survey at the station $\odot_1$ and measure in a direct line to the stations $\odot_2$ and $\odot_3$, taking the offsets on both sides as we proceed. Returning to $\odot_2$ we measure the chain-line $\odot_2$ $\odot_6$, taking the distance of station $\odot_4$ from $\odot_3$ in order to tie the second chain-line with the first. From the station $\odot_6$ we proceed to measure to the station $\odot_8$; the distance from $\odot_7$ to $\odot_5$ will tie the third chain-line with the second; and so on.

Tie	160 to $\odot_{10}$.	line $\odot_{10}\odot_9=160$.
25	50	144
From	$\odot_8$ on L.	
	650 to $\odot_9$.	
	500 to $\odot_8$.	160
30	400	140
100	200	40
Tie	160 to $\odot_7$.	line $\odot_7\odot_5=190$
80	50	
From	$\odot_6$ on R.	
30	450 to $\odot_6$.	
90	310 to $\odot_5$.	60
Tie	200 to $\odot_4$.	line $\odot_4\odot_3=220$.
60	150	100
From	$\odot_2$ on L.	
	450 to $\odot_3$.	
Cross	350	fence to outside.
80	200 to $\odot_2$.	90
50	100	94
40	0	98
From	$\odot_1$ go W.	

Observation. After the same manner the streets in a town may be surveyed.

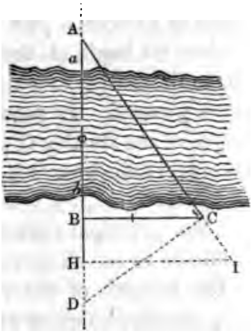
11. Supposing the last figure to represent a river, it is required to draw a plan of it from the following field-book.

Here we begin at the station A, and measure in a direct line to the station B, taking the offsets to the edge of the river as we proceed. In measuring the next chain-line B $\odot_2$, the length of the tie-line $\odot_1$ A is taken, to enable us to lay down the two chain-lines AB and BC. The tie-line to the next chain-line is $\odot_3$ $\odot_2$; and so on. It will also be observed that the principal stations A, B, C, D, &c., are taken, so as to enable us to see along the successive bendings of the river.

The breadth of the river may be taken at any point by the methods explained at pp. 24. 49. and 228.

Tie	100 to F.	line F$\odot_6$ = 160.
	0	30
From	E on L.	
	430 to E.	50
	400	90
	320 to $\odot_6$.	80
	150	22
Tie	100 to $\odot_5$.	line $\odot_5\odot_4$ = 90.
	90	20
From	D on R.	
	400 to $\odot_4$.	
	280 to D.	45
	200	30
	150	25
Tie	100 to $\odot_3$.	line $\odot_3\odot_2$ = 110.
	50	40
From	C on R.	
	490 to $\odot_2$.	
	380 to C.	
	280	40
	200	30
Tie	150 to $\odot_1$.	line $\odot_1$ A = 260.
	100	43
	0	40
From	B on L.	
	150 to B.	20
	100	35
	0	40
From	A go W.	

12. To continue a chain-line DB across a river, the line being measured up to B. Set up a flag at A in a line with DB; with the cross-staff, or by Geo. Art. **48.**, lay down BC perpendicular to DB; measure BC (=400 links), and lay down CD perpendicular to CA, intersecting the chain-line in the point D; measure DB (=300 links); then BA may be readily found.



Because DCA is a right angle, we have, by Cor. 1. page 58., DB . BA $= \text{BC}^2$, $\therefore$ BA $= 400^2 \div 300 = 533\frac{1}{3}$ links.

Or thus, when there is some obstruction in the line CD. Measure any line BC (=400 links) along the side of the river; measure off any convenient distance BH (= 300 links); measure HI (=600 links) parallel to BC, and such that I shall be in a straight line with CA.

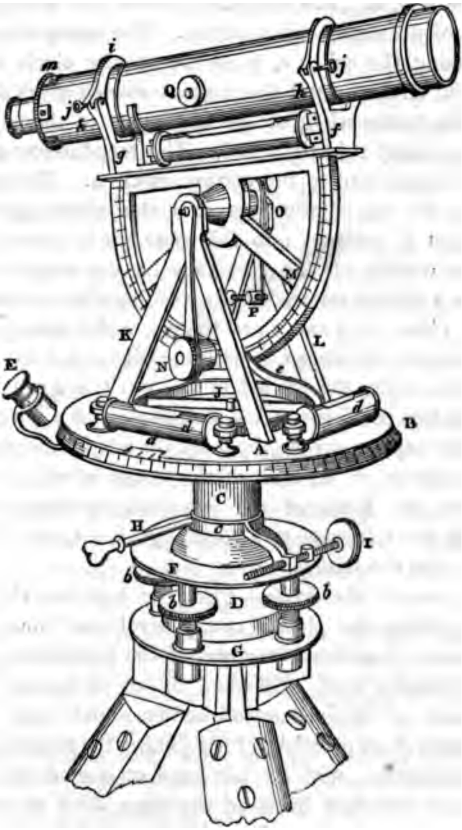
Here from the similar triangles AHI and ABC, we have, HI : BC :: HA : BA, or putting x for BA, 600 : 400 :: $x +$ 300 : x, hence we find x or BA = 600 links.

Other methods are given at pages 24, 49, and 134.

TO SURVEY WITH THE THEODOLITE.

5. The theodolite is the most useful instrument for taking angular measurements in a survey. It consists of a telescope *m i*; two graduated circles B and M, at right angles to each other, divided into degrees and half degrees, and a stand G with three jointed legs for supporting the instrument.

The circle B, having a horizontal motion on the vertical axis C, is used for taking horizontal angles; and M, having a vertical motion on the horizontal axis O, is used for taking *vertical* angles. Besides the graduated plate B, there is another horizontal plate A, called the *vernier circle*, which turns upon a vertical axis passing right through the hollow



axis C, and terminating in the plate G. This vernier circle, therefore, has a motion independent of the graduated circle B. The frame KL stands upon the vernier circle, and carries the axis O of the telescope *mi* and the vertical circle M. The two spirit levels *d* and *d* are fixed at right angles to each other, to enable the surveyor to set the parallel plates A, B, and F, perfectly horizontal by means of the four milled screws *b*, *b*, *b*, *b*. The compass J is used to determine the bearing of the stations in a survey. The clamp-screw H, by compressing the collar *c*, fixes the vernier circle A to the graduated circle B; and the tangent-screw I gives a delicate movement to the whole of the upper part of the instrument when the clamp H is tightened. The tangent-screw P gives a similar movement to the vertical circle M. The plate G is fixed to the top of the legs, and the upper part of the instrument is screwed into this plate; E is a small microscope for reading off the vernier *a* on the vernier plate A, and N is a similar microscope for reading the vernier on the vertical plate M; *g* and *f* are the YS, which simply support the telescope; the clips *i*, *i*, with the pins *j*, *j*, keep the telescope from being thrown out of the YS; Q is a milled screw for adjusting the focus of the object-glass of the telescope; *m* are four capstan screws, by means of which the instrument maker adjusts, if necessary, the cross wires, which are placed in the focus of the eye-glass for the purpose of directing the telescope to a point; *gf* is a spirit level suspended from the telescope.

6. *To use the theodolite.*—Push the legs into the ground so as to bring the plate A nearly level, and the plummet, which hangs from the lower part of the instrument, directly over the station staff. Unclamp the screw H, and turn the upper part of the instrument bodily round until the two levels *d* and *d* are parallel to lines joining the screws *b*, *b*, *b*, *b*, taken diagonally; with the left hand on one of the diagonal screws and the right hand on the other, turn them in *contrary* directions until the level parallel to them is adjusted; then proceed with the other two screws in the same manner; when this is done the first two screws will probably require retouching. Having thus got the plate A perfectly level, turn it until the broad arrow on the vernier coincides with *the zero point* in the circumference of the circle B; turn

both plates now bodily round until the telescope nearly bears upon the left station; clamp the two plates by tightening the clamp-screw H, and then move the tangent-screw I until the centre of the cross wires exactly bears upon the station or object; now unclamp the screw H, and move the upper plate from left to right, by means of a milled screw not shown in the cut, until the telescope exactly bears upon the other station; apply the microscope to the vernier and read off the angle, and it will be the horizontal angle between the two stations required. As there are two verniers upon the plate A to enable the surveyor to make corrections for errors of graduation and centering, the second vernier should also be read off, and the mean of the two readings will be the corrected angle.

When a vertical angle is to be taken, turn the vertical plate M until the bubble of the spirit-level *gf* is exactly in the centre, or, what is the same thing, until the broad arrow of the vernier, observed by the microscope N, is exactly on the zero point of the circumference of the vertical circle; turn the vertical circle until the telescope exactly bears upon the top of the object, and then read off the vernier, and it will give the vertical angle required.*

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* ADJUSTMENTS.

Before using the theodolite there are certain adjustments which should be attended to; the following are the most essential: —

1. To adjust the line of collimation, or to make the intersection of the cross wires in a line with the centres of the eye-glass and object-glass. Direct the telescope to some well-defined distant object; turn the telescope within the YS, and observe whether, during the whole revolution, the centre of the wires always covers the same point: if it does, the instrument is in this respect perfect; but if it does not, correct for half the error by means of the capstan-headed screws *m*, *m*, &c., loosening one screw and tightening its opposite one.

2. To make the axis of the level *gf* parallel to the axis of the telescope. Bring the level *gf* attached to the telescope perfectly level by the tangent-screw P; then reverse the telescope in the YS: if the level remains true, it is in adjustment; if it does not, correct for half the error by the adjusting-screw *f*, placed at the extremity of the level, and for the other half by the tangent-screw P. This process must be repeated, until the instrument is in this respect perfect.

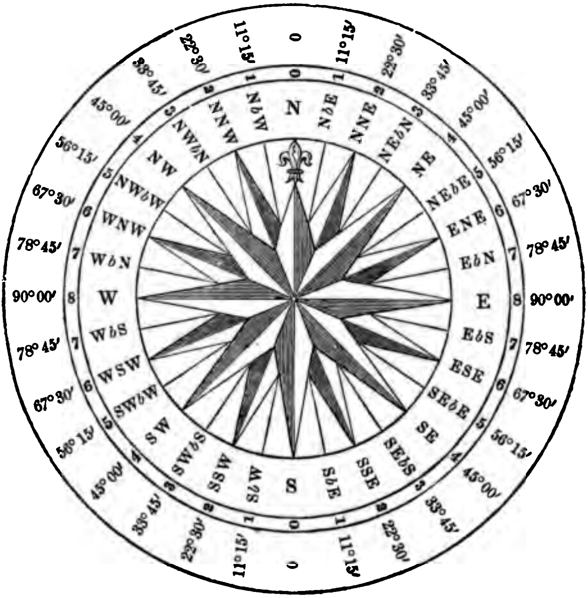
3. To make the axes of the levels, *d*, *d*, on the vernier plate, parallel to

7. *Construction and use of the vernier.*—A vernier is a contrivance for subdividing a graduated scale into any number of parts. In the vernier of the theodolite 29 half degrees on the plate B are divided into 30 equal parts on the vernier *a*. Hence each division of the former will be 30′, while each division of the latter will be $\frac{1}{30}$ of 29 half degrees or 29′; therefore the difference between each will be 1′; and 2 divisions on the vernier will be 2′ less than 2 divisions on the plate B; 3 divisions on the one will be 3′ less than 3 divisions on the other; and so on. Let us suppose, for example, that the arrow of the vernier stands a little past 25° on the lower plate, and that the 6th division of the vernier exactly coincides with a mark on the lower plate, then the 6 divisions on the vernier will fall 6′ short of the 6 divisions on the lower plate; that is, the broad arrow will be 6′ past the 25°, and the angle, therefore, will read 25° 6′. This shows that in reading off the vernier, we first ascertain the number of degrees, or degrees and a half, beyond which the arrow stands, and then for the number of minutes we count the number of divisions on the vernier until we come to where the marks on the two plates coincide, and this number will give the minutes in the angle.

it. Bring the bubbles to the centre of their respective levels; turn the vernier plate half round, and if the bubbles do not remain in their centres, correct one half the error by the screws attached to the levels, and the other half by the screws *b*, *b*, &c. This process must be repeated until the bubbles are found to remain central in every position of the vernier plate.

4. To ascertain whether the vertical arc moves in a vertical plane. Set the plate A perfectly level; direct the telescope to some vertical angle of a building; turn the vertical arc, and observe whether the intersection of the cross wires covers the vertical angle throughout the whole of the motion; if it does, the adjustment is perfect. Any imperfection in this adjustment can only be remedied by an instrument maker. This observation will also apply to some other delicate adjustments, which need not be described in this work.

The Compass.



8. The method of taking the bearings of objects by the compass, depends upon the remarkable physical fact that a magnetised needle placed upon a fine centre will, in any given place, always point in the same direction, however much we may shift the compass-box. That end of the needle which points towards the north is called the north pole, and that which points towards the south the south pole. The north end of the needle, in this country, points at present about 24° to the west of the true north point. This is called the deviation of the magnetic meridian; and it varies not only with respect to the place upon the earth's surface, but also with respect to the period of observation.

In order to find the bearing of an object by the theodolite, place the arrow of the vernier plate at zero; turn the upper plate round until the needle points due N and S; read off the angle swept over by the vernier, and it will give the angle which a line drawn to the object forms with the magnetic meridian.

Examples.

1. To draw a plan of a river from the following field notes of a survey taken with the theodolite. (See fig. 1. to Ex. 10. p. 226.)

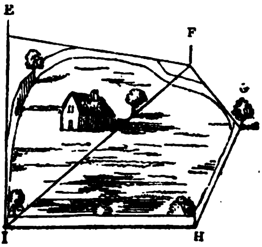
Station-flags are set up at the stations A and B, and the line AB with its offsets are measured. Having arrived at B, a station-flag is placed at the next station C; with the theodolite on the station B the angle ABC is taken, and entered in the field-book as "∠ between A and C." Having arrived at C, a station-flag is placed at D, and the angle BCD is taken; and so on. With the theodolite at C, the angle formed by the stations A and E may be taken as a *check angle* to test the accuracy of the survey. In surveys of this kind tie-lines are not required.

	100 to F.	30
	0	30
∠ bet. D	104°	and F.
From	E on L.	
	430 to E.	50
	400	90
	150	22
∠ bet. C	133°	and E.
From	D on R.	

	280 to D.	40
	200	30
∠ bet. B	115°	and D.
From	C on R.	
	380 to C.	90
	200	30
	0	40
∠ bet. A	113°	and C.
From	B on L.	
	150 to B.	20
	0	40
From	A go W.	

Construction. — From a scale of equal parts draw AB = 150, and mark off the offsets upon it, thereby giving the line of the river; with the centre of the protractor at B, and one of its edges lying along AB, mark off the angle ABC = 113°; take BC = 380, and mark off the offsets upon it; with the centre of the protractor at C, draw CD, making the angle BCD = 115°; and so on to the remaining chain-lines in the field-book.

To find the breadth of a river by the theodolite, see Ex. 4. page 134.



2. Draw a plan and find the area of a field from the following field-book. *Ans.* 1 *r.* 18 *p.*

Here, commencing at E, all the sides of the field are measured by the chain, together with the insets from the sides EF and IE. Placing the theodolite at I, the angles HIF and FIE are determined. These dimensions are sufficient to enable us to construct the figure; but in order to verify the accuracy of the survey, the angle EFG is taken as a check angle.

	200 to E.	
	140	12
	120	10
	80	0
∠ bet. F	50°	and E.
∠ bet. H	40°	and F.
From	I on R.	
0	200 to I.	0
From	H on R.	

0	110 to H.	0
From	G on R.	
	80 to G.	0
Check ∠ bet. E	135°	and G.
From	F on R.	
	200 to F.	0
	70	0
	40	10
	30	25
From	E. go S.E.	

Construction.—From a scale of 2 inches to 100 links take IH = 200. Place the centre of the protractor at I, with its edge on IH, and mark off the angles 40° and 50°, which will give the position of the lines IF and IE. Take off IE = 200, and on E as a centre, with the radius EF = 200, describe an arc cutting IF in the point F; join EF; on F as a centre, with the radius FG = 80, describe an arc, and on H as a centre, with the radius HG = 110, describe an arc cutting the former in G; join FG and HG; mark off the insets on the lines IF and EF, and this will complete the plan of the field. Apply the protractor to the angle EFG, and if it measures 135°, this will show that the dimensions have been correctly taken. From E and H let fall perpendiculars upon IF; join HF, and let fall a perpendicular upon it from G. These perpendiculars being measured by the scale will be found to be 152·5, 128, and 45 links; and the diagonals IF and HF will be found to be 257·5 and 166 links; whence the area of the figure EFGHI may be readily determined. From this area must be subtracted the area of the inset spaces to obtain the area of the field.

3. Draw a plan and find the area of a pentangular field, ABCDE, from the following field-book.

Ans. 66*a.* 2*r.* 24*p.*

	2741 to E.	
∠ bet. C	91° 20′	and E.
From	D on R.	
	2125 to D.	
∠ bet. B	117° 45′	and D.
From	C on R.	

	1555 to C.	
∠ bet. A	110° 30′	and C.
From	B on R.	
	1940 to B.	
From	A go N.W.	

Here, after constructing the outline of the figure by the scale and protractor, the space may be divided into triangles by drawing the diagonals EB and EC, and then the areas of these triangles can be found by measuring off their bases and perpendiculars.

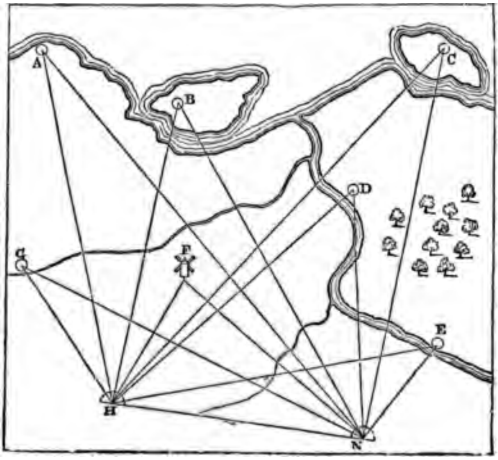
Observation.—To survey a field from the angles taken at one station, see Ex. 5. p. 110. Also, to survey a field from the angle taken at the intersection of the diagonals, see Ex. 1. page 111.

4. In Ex. 14. page 29. let ACDB represent the boundary of a field; it is required to draw a plan of the field, and to find its area.

In this example the side CD of the field is measured as a base line; and from the stations C and D all the angles which the corners make with each other are taken by the theodolite. The plan being plotted as at page 29.; from C and B let fall perpendiculars upon AD; then, from the scale, measure off these perpendiculars, together with the diagonal AD; whence the area will be found to be 5*a.* 1*r.* 18*p.*

TRIANGULATION.

9. A country may be surveyed, and all the distances of towns and important places from each other accurately ascertained, by the method of triangulation, which consists in taking the dimensions of a series of triangles constructed on each other, and running through the district to be surveyed. This may be accomplished by the chain only; but in very extensive surveys it should be done by the chain and theodolite.



Let it be required to give a map of the country of which A, B, C, D, &c. are towns and important places. The two stations H and N are selected for the purpose of observing the objects A, B, C, &c., included in the survey. The base line HN is then accurately measured, and its bearing ascertained by the compass. With the theodolite at H and N all the angles which the objects A, B, C, &c., make with H and N are determined. Hence the triangles HNA, HNB, &c., may be constructed, there being given in each triangle the base and the angles at the base. Moreover the distance between any two of the objects, as A and C, may be found (see Ex. 14. p. 29.). The line AC may now be taken as a base for a new series of triangles; and so on to any extent.

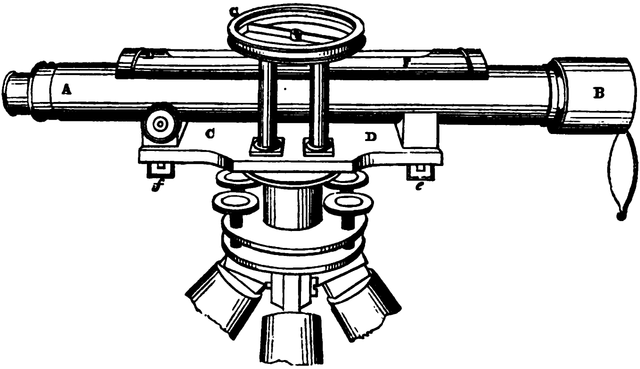
When the survey is made with the chain only, the three sides of all the triangles, forming the system of triangulation, must be measured. In the course of these measurements, the offsets to any particular lines or objects should be entered in the field-book.

Surveys of this kind are first made in order to determine the proper course of a railway line, and to fix piquets at the different points through which it must pass. In filling up the survey, the ground lying 100 yds. or 454 links (at least) on each side of the line must be measured and mapped, after the methods described in the preceding problems.

LEVELLING.

10. A level line is that line which water assumes when in a state of rest. If we assume the earth to be a perfect sphere, which it is very nearly, a level line will be everywhere at the same distance from the centre of the earth. Levelling, therefore, is that branch of surveying, which determines the inequalities of the earth's surface, and ascertains the relative heights of places above a certain line, *called* the datum line, equidistant from the centre of the earth. The line which is determined by the spirit level is, consequently, a tangent to the earth's surface. In finding the levels for buildings, or the drains of fields and commons, the ordinary spirit levels, described at page 7., mounted on a tripod stand, will be found sufficiently accurate; but in long surveys, such as occur in the construction of a railway, more perfect instruments are absolutely required.

Description of Troughton's Level.

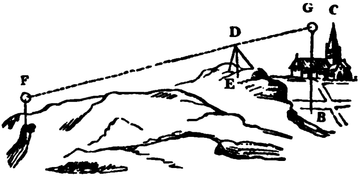


11. In this cut EF represents the spirit-level fixed upon the top of the telescope AB, which rests upon the horizontal plate CD. This plate turns upon a vertical axis, in the same manner as the vernier plate of the theodolite; the compass-box G is supported by pillars passing from the plate CD. There are two parallel plates, and two pairs of conjugate screws for bringing the instrument level, precisely after the manner described for the theodolite. To use the instrument, fix it as nearly level as can be judged by the eye; set the telescope over one pair of the diagonal or conjugate screws, and by means of this pair of screws bring the bubble to the middle of the level; turn the telescope over the other pair

of screws, and level it in this position by means of the diagonal screws; turn it back to the first pair, and correct again if necessary; then if the bubble stands in the middle of the level, in all positions of the telescope*, the instrument is adjusted for use. Turn the telescope towards the levelling staves, and read off the number, &c.

12. *Levelling staves* are rods about twelve feet long, divided into feet, tenths, and hundredths of a foot; these divisions are very distinctly marked, so that the surveyor is able to read off the point, where the intersection of the cross wires of the telescope cuts the staff.

13. To ascertain whether there is a proper descent for water, from a well at F to a village at B. Place the spirit-level D on the eminence E, from which both the levelling staves, F and GB, can be seen; adjust the level, and direct the telescope to the staff F; turn the telescope upon its vertical axis, and direct it to the staff GB; then the difference between the heights of the staves, GB and F, will give the descent of the water from the well to the place B. Suppose the distance from B to F to be 300 ft.; the length of the staff F to be 5 ft., and BG 20 ft.; then the total descent from the well to B will be 20 ft. — 5 ft. = 15 ft.; now as this descent takes place for 300 ft., the descent for 1 ft. will be $\frac{1}{300}$ of 15 ft. = ·05 ft., and for every



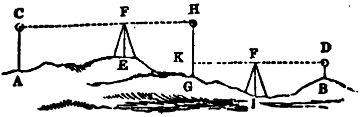
* ADJUSTMENTS.

Should this not be the case, the error must be occasioned by the axis of the level not being truly perpendicular to the vertical axis of the instrument. To correct this error, raise or depress, as the case may require, the capstan screws *e* or *f*, till the bubble be brought half way to the middle of the tube, then correct for the other half by the diagonal screws. Repeat the process until perfect accuracy is attained.

There are two other adjustments, which are precisely the same as the 1st and 2nd given for the theodolite. It is obvious, that these two *adjustments* should be made before the one above described.

100 ft. it will be 5 ft. It is evident from this calculation, that the height of the level does not at all affect the result.

14. To determine the difference of level between two points A and B, situated in a farm, or any extensive piece of land.



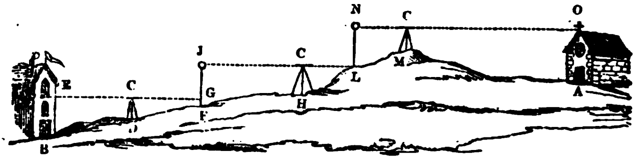
Here DK and HC are supposed to be two level lines taken with the instrument at I and E. When the surveyor goes from A to B, the height GH is called the fore sight and AC the back sight; in like manner BD is the fore sight and GK the back sight. The difference between BD and GK, will give the difference of level between the points B and G; and the difference between GH and AC, the difference of level between G and A; and so on. Let $BD = 4$, $GK = 3$, $GH = 9$, and $AC = 7$ feet; then the rise from B to G $= 4 - 3 = 1$ ft.; the rise from G to A $= 9 - 7 = 2$ ft.; therefore the total rise from B to A $= 1 + 2 = 3$ ft. Again, let $BD = 2$, $GK = 5$, $GH = 12$, and $AC = 8$ feet; then as B is higher than G, we have height of B above G $= 5 - 2 = 3$ ft.; and height of A above G $= 12 - 8 = 4$ ft.; therefore height of A above B $= 4 - 3 = 1$ ft.

Let us now investigate a general rule for ascertaining the difference of level between the points A and B.

The distance of A from the level line HC is AC, and the distance of B from the same line is $HK + DB$; therefore the difference of level between A and B will be expressed by $HK + DB - AC$; adding and subtracting KG, this difference becomes $HG + DB - (AC + KG)$. But HG and DB are the back sights, and AC and KG are the fore sights; hence we have the following general rule: *take the difference of the sums of the back and fore sights or readings, and it will give the difference of levels between the first and last staves.* Thus, in the last example, we have the sum of the back sights $= 2 + 12 = 14$, and the sum of the fore sights $= 5 +$

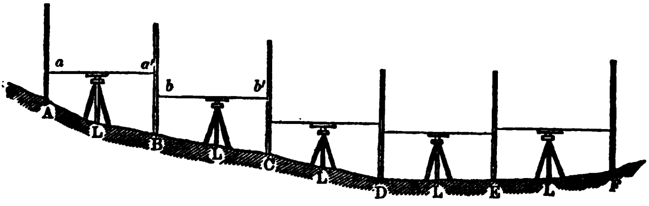
$8 = 13$; therefore difference of level between A and B $= 14 - 13 = 1$ ft.; and as the sum of back sights is greater than the sum of the fore sights, this difference shows that A is 1 ft. higher than B.

15. To determine the difference of level between two houses BE and AO, by a series of sights.



Here we suppose the level lines, or lines of sight, to be taken in order, EG, JL, and NO: hence the back sights are BE, JF, and NL, and the fore sights FG, 0, and AO; then by Art. **14.**, the sum of the latter subtracted from the sum of the former will give the height of the ground at A above B. For example, let the back sights be 9, 11, and 13 feet respectively, and the fore sights 2, 0, and 16 feet; then we find the rise of A above $B = 33 - 18 = 15$ ft.

16. In long surveys, where great accuracy is required, such as in railway levelling, the level L should always be



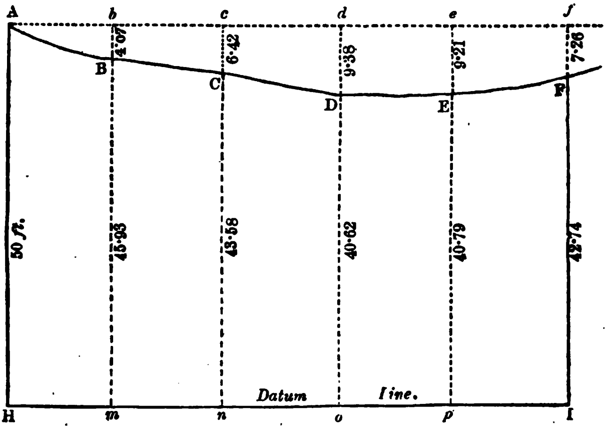
placed in the centre between the staves A, B, C, &c., for when *this is the* case there is no correction required for curvature *and refraction*. In order to give a section of the earth's surface represented by A B C D, &c., place the level in the middle between the staves A and B; after making the spirit-level horizontal, turn the telescope towards the back staff A, and read off the height A*a* at which the horizontal line of sight cuts the staff; this height must be entered in the column of "back sights" of the field-book. Turn the telescope round, and direct it towards the fore staff B, and read off the height B*a′*, which must be entered in the column of "fore sights" of the field-book. Keeping the staff B in the same position, excepting that its graduated side must now be turned towards C, let the staff A be removed to C, and the level to a place in the middle between B and C; proceed as before, entering the height B*b* in the column of "back sights," and C*b′* in the column of "fore sights;" and so on to any extent. As the survey proceeds *intermediate* levels must be taken to *bench marks*, that is, to certain marks made on some fixed and well-defined objects which occur in the line of survey; the sights or levels taken to such marks are entered in the column called *inter.* or intermediate sights. Thus in taking the following field-book:

Field-book.

Back Sights.	Inter.	Fore Sights.	Rise.	Fall.	Reduced Levels.	Distance in Chains.	Remarks.
					50 D.		
4·71	—	8·78	—	4·07	45·93	1·00	B. M. at A.
6·26	—	8·61	—	2·35	43·58	2·00	
5·34	—	8·30	—	2·96	40·62	3·00	
7·33	7·16	—	0·17	—	40·79	3·90	centre of — road.
—	—	5·21	2·12	—	42·74	5·00	
23·64		30·90			7·26		
		23·64					
		7:26 ft.					

The first back sight is 4·71, the fore sight 8·78; the second back sight is 6·26, the fore sight 8·61; the third back sight is 5·34, the fore sight 8·3; then an *inter.* sight is taken, by removing the back staff C to the bench mark at E, and the instrument to a place somewhere between D and F, (see next fig.) the back staff D is then read off 7·33, the *inter.* staff 7·16, and the fore staff F 5·21 feet. The numbers entered in the columns of "rise" and "fall" are obtained by taking the difference of the back and fore sights, observing that when the fore sight is greater than the back sight, the difference indicates a *fall* in the ground, and *vice versâ.* In the column of "reduced levels," 50 D. indicates that the line HI, called the *datum line*, is assumed to be 50 ft. below the first station A; the succeeding numbers indicate the distance of the corresponding station from the datum line; these numbers are consequently found by subtracting the fall from the foregoing reduced level, or adding the rise to it, as the case may be. The numbers in the column of "distance" are the respective distances of the different stations from the first station A. The symbol B.M., which occurs in the column of "remarks," reads bench mark. Adding together the columns of back and fore sights, and taking the difference of the two sums, we obtain 7·26 ft. for the difference of levels of the first and last stations A and F, (see **Art. 14.**); the same thing may be derived from the column of reduced levels, by merely subtracting the last number from the first; this serves as a check to the accuracy of the calculation.

Construction. Draw HI for the datum line. With a scale of about an inch to a chain lay off the horizontal distances of the stations, viz., Hm = 1, Hn = 2, Ho = 3, Hp = 3·9, and HI = 5. Draw HA, mb, nc, &c., perpendicular to HI; from a scale of an inch to 25 ft. mark off the "reduced levels" HA, mB, nC, &c., then ABCDEF will represent the line of the ground. Here the vertical dimensions are taken on a much larger scale than the horizontal ones, *otherwise* the inequalities of the ground would, in most cases, be scarcely perceptible. The datum line should be chosen, so as to be below the ground line at every point.



Sectional Drawing from the foregoing Field-book.

To survey the ground for a railway.

17. *Trial levels.* Aided by a model map, giving the physical features of the country, the surveyor first marks out the route of the projected line on an Ordnance map. Commencing the survey at one of the termini, he takes the levels, as nearly as possible in the direction marked out on the map. In the course of the survey, the heights of roads, crossed by the line, are carefully taken, and also bench marks near such points; cross-levels are also frequently taken of such roads to show the inclination of the ground on each side of the line. Should the line, thus surveyed, be found ineligible, on account of some engineering difficulties or other causes, another survey must be made, embracing the whole, or only a portion, of the distance between the termini.

Check levels. The trial levels having been found satisfactory, the whole surface of the ground, from 5 to 10 or 15 chains on each side of the projected line, must then be surveyed; this survey must include every field, building, road, &c., within these limits. It is usual to plot the plan of the ground on a scale of about 8 chains to an inch, while the vertical lines in the section are drawn on a scale of about 25 ft. to an inch. For further details relative to parliamentary plans and sections, the student may consult *Mr. Baker's* edition of "Nesbit's Surveying."

Corrections for curvature and refraction.

18. It has been shown, at page 59., that the correction for curvature is 8 inches for every mile, and moreover that the correction varies with the square of the distance: thus the correction for 2 miles will be 4 times 8 in. = 32 in.; for 3 miles it will be 9 times 8 in. = 6 ft.; and so on. Hence we have the following rule: *multiply the square of the distance in miles by* 8, *and it will give the correction for curvature, in inches.*

The refraction of the atmosphere tends to make objects appear higher than they really are. When the air is not overloaded with moisture the correction for refraction may be taken at $\frac{1}{7}$ of that which is due to curvature. This correction must obviously be deducted from the correction for curvature, in order to obtain the total correction.

In the case of levelling, *the total correction must be subtracted from the observed reading to obtain the true reading;* or the correction may be added to the height of the instrument, and then the difference between this sum and the reading of the staff will give the difference of level between the two stations. When the elevation of objects above the horizon is taken by the theodolite, the correction, as well as *the* height of the instrument, must be *added* to the *calculated height* to obtain the true height.

When the spirit level is placed equidistant from the two station staves, no correction is required; because the correction for each staff being the same, the *difference* of the apparent heights or readings must be the true difference of level between the two stations.

1. Required the correction for curvature, when the distance of the object is 1 chain.

cor. curv. for 1 mile or 5280 ft. $= 8$ in.

∴ cor. „ 1 chain or 66 ft. $= 8 \times (\frac{66}{5280})^2 = \frac{1}{800}$ in.

Hence we have the following rule: *take* $\frac{1}{800}$ *of the square of the distance in chains, and it will give the correction for curvature in inches.*

2. Required the correction for curvature and refraction, when the distance of the object is $1\frac{1}{8}$ miles.

$$\begin{aligned} \text{cor. for curv.} &= 8 \times (\tfrac{9}{8})^2 = 10{\cdot}125 \\ \therefore \text{cor. for refr.} &= \tfrac{1}{7} \text{ of } 10{\cdot}125 = \underline{1{\cdot}446} \\ \therefore \text{Total correction} &= \underline{8{\cdot}678} \text{ in.} \end{aligned}$$

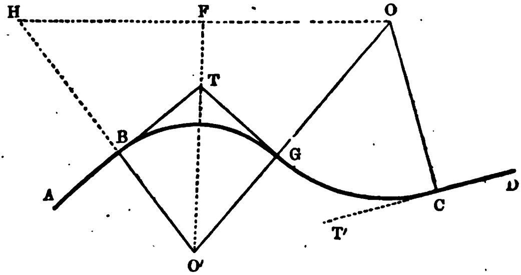
3. Required the correction for curv. and refr., when the distance of the object is 20 chains.

$$\begin{aligned} \text{cor. for curv.} &= \tfrac{1}{800} \times 20^2 = {\cdot}5 \\ \text{cor. for refr.} &= \tfrac{1}{7} \text{ of } {\cdot}5 = \underline{{\cdot}0714} \\ \therefore \text{Total correction} &= \underline{{\cdot}4285} \text{ inches.} \end{aligned}$$

RAILWAY CURVES, ETC.

19. A railway should be made as straight as possible; but owing to the obstructions that are unavoidably met with, and the expediency of turning the rail so as to pass through, or near to, places of importance, the engineer frequently finds it most economical to connect different portions of the line by what are called railway curves, which are always arcs of circles. On account of the danger there is of the carriages running off rails of a great curvature, no railway curve is allowed, by government, to have a radius less than 80 chains or 1 mile. The nature of the curve, connecting two straight portions of rail, depends in general upon the angle which they make with each other. In every case the straight portions must form tangents to the connecting curve.

20. When the two lines of rail, AB and CD, form a small angle with each other, the connecting curve BGC must have a serpentine form, or what is called the S curve.



1. To construct the S curve, when the points of contact B and C are given, and also the radius OC of one portion of the curve CG.

Draw BH perpendicular to AB, and CO perpendicular to CD. Take CO and BH respectively equal to the given radius; join OH, and bisect it in F by the perpendicular FO′, cutting HB produced in O′; then O′ will be the centre of the curve BG; join O′O; with O as a centre and the radius OC describe the arc CG, and with O′ as a centre and the radius O′G describe the arc GB.

Because the triangles O′HF and O′OF are identical, O′H = O′O; but HB = CO = OG; therefore O′B = O′G, and (Geo. Art. 62.), the curves CG and BG will touch each other in the point G.

2. To construct the S curve, when the radius CO is given, and the arcs CG and BG have equal radii.

From *any* point B in the line AB draw BO′ perpendicular to AB; take BO′ = CO, and through O′ draw a line (not *shown in* the fig.) parallel to AB; on O as a centre with a *radius* equal to 2 CO, describe an arc cutting this parallel line in a point which we shall call O′, then this point will be the centre of the curve BG.

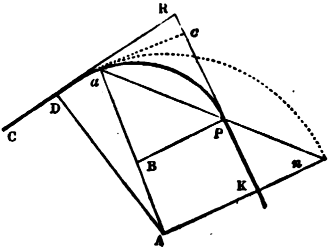
In a similar manner other cases of this problem may be constructed.

21. When the two lines of rail form a considerable angle with each other, the connecting curve must be concave towards the same side.

1. Let AB and TG be the direction of two rails, (see last fig.); and let the point of contact B be given, to find the other point of contact G, &c.

Produce the directions of the two rails, until they cut each other in the point T; draw TO′ bisecting the angle BTG, and cutting the perpendicular BO′ in the point O′; from O′ draw O′G perpendicular to TG; then O′ will be the centre of the connecting curve BG, and G will be the point of contact required. For the triangles O′BT and O′GT are obviously identical, and therefore O′G = O′B, &c.

2. To construct the double curve D*a*P, connecting the two rails CD and KP, when the two points of contact D and P are fixed, and the radius AD, of the portion D*a*, is given. With the centre A and radius AD describe the arc D*an*; draw AK*n* perpendicular to PK cutting the arc D*an* in the point *n*; join *n*P, and produce it cutting the arc D*an* in the point *a*; join A*a*, and from P draw PB parallel to AK; then B will be the centre of the arc *a*P, the remaining portion of the connecting curve.



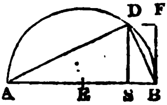
For the triangles A*na* and BP*a* being similar, and A*n* = A*a*, we have BP = B*a*; moreover because AK is perpendicular to PK, therefore BP, which is parallel to AK, will also be perpendicular to PK; hence PK will be a tangent to the curve *a*P.

3. To construct the double curve D*a*P, when the point of contact D, and the radius AD, are given.

With the centre A and radius AD, describe the arc D*a* of any convenient length; join A*a*, and draw *ac* perpendicular to A*a*, cutting KR in *c*; take *c*P $=$ *ca*, and draw PB perpendicular to KP, cutting A*a* in the point B; then B will be the centre of the arc *a*P.

To lay down a railway curve on the ground.

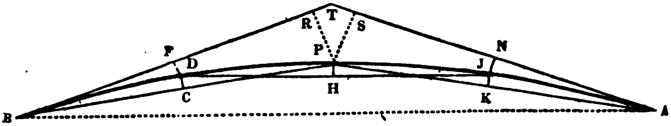
22. Let AB be the diameter of the circle BDA; BD the chord of the arc BD; BF a tangent to the point B; DS perpendicular to AB; and DF perpendicular to BF; then because ADB is a right angle (Cor. 2. p. 55.), the triangles ABD and DBS are similar,



$$\therefore \text{AB} : \text{BD} :: \text{BD} : \text{BS or DF} = \frac{\text{BD}^2}{\text{AB}} \ldots (1).$$

Now in the case of railway curves, the chord BD is very small as compared with the diameter; hence the chord BD may be taken as the length of the arc without incurring any sensible error; but in setting out these curves, BD is usually taken equal to 1 chain; putting r therefore for the radius of the circle, we have BS or DF $= \frac{1}{2r}$. Let r, for example, be 80 chains; then DF $= \frac{1}{2 \times 80}$ chains $=$ 4·95 inches.

1*st method.*—Let BF and AN be the directions of two portions of a railway. It is required to lay down the connecting curve BDPJA, the radius being 80 chains.



Along the direction of the rail, measure BF $=$ 1 chain, and *take* FD $=$ 4·95 in.; then D will be a point in the curve.

Take DC = 4·95 in., and measure BP = 2 chains in a line with BC; then P will be a point in the curve. Take PH = 4·95 in., and measure DJ = 2 chains in a line with DH; then J will be a point in the curve. And so on to any extent.

2nd method.—On the tangent line BT, take BF = 1 chain, BR = 2 chains, and so on; then there will be a certain portion RT remaining, which must be measured. Here the 1st perpendicular FD $= \frac{1}{160}$ ch. = 4·95 in.; by the preceding general expression (1), the 2nd perpendicular RP $= \frac{BR^2}{2r} = \frac{2^2}{160}$ ch. $= 4 \times 4{\cdot}95$ in. = 19·8 in.; similarly the 3rd perpendicular $= \frac{3^2}{160}$ ch. $= 9 \times 4{\cdot}95$ in. = 3 ft. 8 in.; and so on. Hence we have the following construction: At the distance of 1 chain from B, take FD = 4·95 in.; then D will be a point in the curve; at the distance of 2 chains from B, take RP = 19·8 in., then P will be a point in the curve; and so on. Along the tangent TA measure TS = TR, and proceed to mark off the perpendiculars on TA, in a reverse order to that in which they were taken on BT.

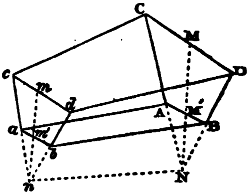
3rd method.—The curve having been first constructed on paper, Art. **19.**, the angle BTA will be known; let it, for example, be 140°; then $\angle$ ABT or $\angle$ BAT $= \frac{1}{2}(180° - 140°) = 20°$. Let the curve be divided into any convenient number of equal parts, for example let the number of parts be four, then (Geo. Art. **60.**, Cors. 1. and 3.), $\angle$ DBF = $\angle$ PBD = $\angle$ JBP = $\angle$ ABJ $= \frac{1}{4}$ of $20° = 5°$. And the angles formed at the extremity A will have the same relation.

Let theodolites be placed at B and A, then by taking the $\angle$ DBF $(= 5°)$ and $\angle$ DAN $(= 3 \times 5° = 15°)$ at the same time, the point D is determined. In like manner, by taking $\angle$ PBF $= 2 \times 5° = 10°$, and $\angle$ PAN $= 2 \times 5° = 10°$, the point P is determined; and so on to the remaining points in the curve.

These methods admit of obvious modifications to suit particular circumstances. Mr. Baker, I believe, was the first mathematician that published anything on this subject. See his little work on railway engineering.

RAILWAY CUTTINGS AND EMBANKMENTS.

23. Let the plane of the ground DC*cd* be parallel to the section DA*ab* of the rail.



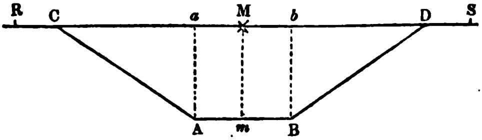
1. *To set out the width of the ground.*

Putting $g =$ AB or ab, the breadth of the rail; $h_1 =$ MM′, the height of the section ABDC; p to 1 the ratio of the slopes; then we have (by Art. **28.**, p. 181.) MD or MC $= \frac{1}{2}(g + 2p\,h_1)$; and, in like manner, putting $h_2 = mm'$, the height of the section $abdc$, we have md or $mc = \frac{1}{2}(g + 2p\,h_2)$. These two expressions give the width of the ground, at the top of the cutting, measured on each side of the central stumps M and m.

Ex. If the bottom width AB = 30 ft., the depth 20 ft., and the slopes $1\frac{1}{2}$ to 1; required MD or MC.

Here $g = 30$, $h_1 = 20$, and $p = 1\frac{1}{2}$; hence we have MD or MC $= \frac{1}{2}(30 + 2 \times 1\frac{1}{2} \times 20) = 45$ ft.

Construction. Take AB = 30, bisect it in the point m; draw mM, Bb, and Aa, perpendicular to AB; take mM = 20, and draw RMS parallel to AB; take bD and aC, respectively, *$1\frac{1}{2}$ times* mM $= 1\frac{1}{2} \times 20 = 30$; join BD and AC; then ABDC will represent the cutting, and MD or MC = 45 ft., will be the



half width required. If DS = CR = 8 ft., be the distance of the fences from the edge of the cutting, then the total width of the ground will be RS $= 2 \times 45 + 2 \times 8 = 106$ ft.

2. *To find the area of the ground.*

The area of the ground between the sections, having the form of a trapezoid, may be calculated by the rule of Art. **5.**, p. 146.

Ex. Required the area of the ground of *Ex.* 3., p. 183.

Here the top widths are 64 and 72 ft. respectively; and the perpendicular distance between them 132 ft.

$\therefore$ area $= \frac{1}{2}(64 + 72) \times 132 = 8976$ sq. ft. $= 997\frac{1}{3}$ sq. yds.

When there are a series of equidistant sections, their area will be more expeditiously found by the following formula:—

Substituting the top widths, Art. **28.**, p. 181, in the formula of Art. **16.**, p. 160, and reducing, we find the area of the ground included by n equidistant sections taken l chains apart to be,

$$\frac{l}{660}\left[(n-1)g + p\{h_1 + h_n + 2(h_2 + h_3 + \ldots + h_{n-1})\}\right] \text{acres.}$$

Ex. Required the area of the ground occupied by a railway, when the heights of the sections taken at 1 chain apart, are 4, 6, 10, 12, and 0 ft.; the slopes being 2 to 1; and the width of the rail 30 ft.

Here $l = 1$, $n = 5$, $g = 30$, $p = 2$, $h_1 = 4$, $h_2 = 6$, $h_3 = 10$, $h_4 = 12$, and $h_5 = 0$; then

$$\text{area} = \frac{1}{660}\left[4 \times 30 + 2\{4 + 0 + 2(6 + 10 + 12)\}\right] = 1\,r.\,18{\cdot}18\,\text{p.}$$

3. *To find the content of the cutting, &c.*

Substituting the areas of the end and middle sections, given in Art. **28.**, p. 181, in the prismoidal formula, and reducing we find,

$$\text{Solidity ABC}dab = \frac{66l}{6}\left[(g+ph_1)h_1 + (g+ph_2)h_2 + 4\left\{g+\frac{p}{2}(h_1+h_2)\right\}\tfrac{1}{2}(h_1+h_2)\right] \text{ c. ft.}$$

$$= lg \,.\, \tfrac{11}{9} \,.\, (h_1+h_2) + lp \,.\, \tfrac{22}{27}\{(h_1+h_2)^2 - h_1h_2\} \text{ c. yds.}$$

Ex. Required the solidity of the cutting of Ex. 3., p. 183. by the preceding formula.

Here $l=2$, $h_1=8$, $h_2=10$, $g=32$, and $p=2$;

$$\therefore \text{Solidity} = 2\times 32\times \tfrac{11}{9}(8+10) + 2\times 2\times \tfrac{22}{27}\{(8+10)^2 - 8\times 10\}$$
$$= 2203{\cdot}26 \text{ cubic yards.}$$

Practical men usually calculate cuttings and embankments by means of tables, giving the solidity corresponding to the given heights. In the preceding formula, the first part of the expression is given in Bidder's tables as the "central part," l and g being unity; and the second part of the expression, as the "slopes," p and l being unity; so that the solidity is obtained, from these tables, by simply multiplying the respective parts by the given values of l, g, and p, after the manner shown in the formula. Some useful tables have also been published in "Baker's Railway Engineering."

24. Let the ground fall or rise, on each side of the central line Mm. (See fig. Art. **23.**)

1. *To set out the width of the ground.*

Let the slope of the ground MD be q_1 feet fall for every horizontal foot, q_2 the slope of md, Q_1 the slope of MC, and Q_2 the slope of mc. From D let fall a perpendicular upon MM′ intersecting this vertical line in a point (not shown in the fig.) which we shall call J; put $x=$ DJ, then we have,

$$\text{M}'\text{B} : \text{NM}' :: p : 1; \quad \therefore \text{NM}' = \frac{g}{2p};$$

$$\therefore \text{NM} = \text{NM}' + \text{M}'\text{M} = \frac{g}{2p} + h_1 = \frac{g+2ph_1}{2p}.$$

$$\text{MJ} : \text{DJ} :: q_1 : 1; \quad \therefore \text{MJ} = q_1 x;$$

$$\therefore \text{NJ} = \text{NM} - \text{MJ} = \frac{g+2ph_1}{2p} - q_1 x.$$

$$\text{DJ} : \text{NJ} :: p : 1; \ \therefore \text{NJ} = \frac{x}{p};$$

$$\therefore \frac{x}{p} = \frac{g+2ph_1}{2p} - q_1 x; \ \therefore x \text{ or DJ} = \frac{g+2ph_1}{2(1+pq_1)}.$$

Hence we have by substitution, $\text{MD} = \sqrt{\text{MJ}^2 + \text{DJ}^2}$

$$= \sqrt{q_1^2 x^2 + x^2} = x\sqrt{q_1^2+1} = \frac{g+2ph_1}{2(1+pq_1)}\sqrt{q_1^2+1}.$$

Where q_1 is plus or minus according as the ground of the cutting falls or rises from the stump M; and the observation will also apply to all the formula hereafter given. This formula also expresses the width *md*, by merely changing h_1 into h_2, and q_1 into q_2.

Ex. 1. Let the height of the cutting be 16 ft.; the width of the rail, 33 ft.; the slope of the side, 2 to 1; and the *fall* of the ground, ½ to 1. Required the half width MD of the ground.

Here $h_1 = 16$, $g = 33$, $p = 2$, and $q_1 = \frac{1}{2}$, then,

$$\text{MD} = \frac{33+2\times2\times16}{2(1+2\times\frac{1}{2})} \times \sqrt{\tfrac{1}{4}+1} = 27{\cdot}11 \text{ ft.}$$

Ex. 2. Required the same as in the last example, when the ground *rises* ¼ to 1.

Here $q_1 = -\frac{1}{4}$, hence we have,

$$\text{MD} = \frac{33+2\times2\times16}{2(1-2\times\frac{1}{4})} \times \sqrt{\tfrac{1}{16}+1} = 99{\cdot}97 \text{ ft.}$$

2. *Solidity of an irregular cutting.*

As the slope of the sides, in some cases, not only depends upon the nature of the material, but also upon the height, we shall suppose P to be the ratio of the slope on the side *ACca*, the other notation remaining the same as in the preceding articles; then the solidity of $\mathrm{ABC}ba$ will be expressed, in cubic yards, by the formula,

$$\frac{11l}{108pr_1r_2^2}\{r_2^2k_1^2+r_1r_2k_2(k_1+k_2)\}$$

$$+\frac{11l}{108\mathrm{P}\mathrm{R}_1\mathrm{R}_2^2}\{\mathrm{R}_2^2\mathrm{K}_1^2+\mathrm{R}_1\mathrm{R}_2\mathrm{K}_2(\mathrm{K}_1+\mathrm{K}_2)\}-\frac{11lg^2}{36}(\mathrm{P}+p).$$

Where $r_1=1+pq_1$, $r_2=1+pq_2$, $\mathrm{R}_1=1+\mathrm{P}\mathrm{Q}_1$, $\mathrm{R}_2=1+\mathrm{P}\mathrm{Q}_2$, $k_1=g+2ph_1$, $k_2=g+2ph_2$, $\mathrm{K}_1=g+2\mathrm{P}h_1$, and $\mathrm{K}_2=g+2\mathrm{P}h_2$.

This very general formula has been derived on the assumption, that the ground, on each side of the line $\mathrm{M}m$, is formed by two planes passing through the edges $md\mathrm{D}$ and $\mathrm{DM}m$, on the one side, and $mc\mathrm{C}$ and $\mathrm{CM}m$, on the other.

When $q_1=q_2$, and $\mathrm{Q}_1=\mathrm{Q}_2$; and $\therefore$ $r_1=r_2$, $\mathrm{R}_1=\mathrm{R}_2$; and moreover putting $a_1=\frac{k_1}{2p}=\frac{g}{2p}+h_1$, $a_2=\frac{k_2}{2p}$, $\mathrm{A}_1=\frac{\mathrm{K}_1}{2\mathrm{P}}$, and $\mathrm{A}_2=\frac{\mathrm{K}_2}{2\mathrm{P}}$; then this formula after reduction becomes,

$$\frac{lp}{2r_1}\cdot\tfrac{22}{27}\{(a_1+a_2)^2-a_1a_2\}+\frac{l\mathrm{P}}{2\mathrm{R}_1}\cdot\tfrac{22}{27}\{(\mathrm{A}_1+\mathrm{A}_2)^2-\mathrm{A}_1\mathrm{A}_2\}$$

$$-\frac{11lg^2}{36}(\mathrm{P}+p)=\frac{lps}{2r_1}+\frac{l\mathrm{P}\mathrm{S}}{2\mathrm{R}_1}-\tfrac{11}{36}lg^2(\mathrm{P}+p).$$

Where s is the solidity of the slopes taken from Bidder's tables for the heights a_1 and a_2, and S for the heights A_1 and A_2.

When $\mathrm{P}=p$, and $\therefore$ $\mathrm{S}=s$, this last formula becomes,

$$\frac{lps}{2}\left(\frac{1}{r_1}-\frac{1}{\mathrm{R}_1}\right)-\tfrac{11}{18}lpg^2.$$

MISCELLANEOUS QUESTIONS.

1, What length must be cut off from a board 13 inches broad to contain $13\frac{1}{2}$ sup. ft.? *Ans.* 12·46 *ft.*

2. A rectangle is 36 ft. long, and 25 ft. broad; required the side of a square which contains the same area. *Ans.* 30 *ft.*

3. Two travellers, A and B, left the same place at the hour of 9 in the morning; A went north-west at the rate of 3 miles an hour, and B north-east at the rate of 5 miles an hour; how far were they apart from each other at noon of the same day? *Ans.* 17·49 *miles.*

4. The sides of three squares are 8, 20, and 12 feet respectively; required the side of a square which is equal in area to all the three. *Ans.* 24·657 *ft.*

5. Required the length of a cord which will describe a circle containing $\frac{3}{8}$ of an acre of ground. *Ans.* 24·036 *yds.*

6. The wheel of a carriage turns round 2000 times in a distance of 5 miles; what is its diameter? *Ans.* 4·201 *ft.*

7. The surface of a cube is 96 sq. ft.; required its side. *Ans.* 4 *ft.*

8. How many 3-inch cubes can be cut out of a rectangular solid 4 ft. long, 2 ft. broad, and 1 ft. thick? *Ans.* 512.

9. The area of a trapezoid is 28 sq. ft., and the two parallel sides are 6 and 8 ft. respectively; required the perpendicular distance between them. *Ans.* 4 *ft.*

10. What is the solidity of a cube, whose diagonal is 15 inches? *Ans.* 649·5 *c. in.*

11. The base of a field in the form of a trapezoid is 15 chains, and the two perpendiculars are 14 and 8 chains respectively; it is required to find the length of the line parallel to the perpendiculars which will divide the field into two equal parts. Ans. 11·4 chains.

12. The height of a cylinder is 40, and its diameter is 10; required the height of another cylinder whose diameter is 30, and solidity double the former. *Ans.* $8\frac{8}{9}$.

13. If the side of a square be increased by 2 feet, the surface will be increased by 36 sq. ft.; required the side. *Ans.* 8 *ft.*

14. If the linear side of a cube be increased 1 ft., the solidity will be increased by 37 c. ft.; required the side. *Ans.* 3 *ft.*

15. What length of wire $\frac{1}{10}$ of an inch in diameter, may be drawn out of a cubic inch of metal? *Ans.* 10·61 *ft.*

16. The diameter of a cylindrical vessel is 5 ft.; what must be its depth so that it may contain 80 gallons, allowing 277·274 c. in. to one gallon? *Ans.* 7·845 *in.*

17. If the pressure of the atmosphere be 15 lbs. upon every superficial inch, what will be its pressure upon the piston of a steam-engine whose diameter is 20 in.? *Ans.* 4712·4 *lbs.*

18. The area of a hexagon is 14 sq. ft.; required the length of its side. *Ans.* 2·3212 *ft.*

19. The area of a circle is 39·27 sq. ft.; required the difference of the areas of the circumscribed and inscribed squares. *Ans.* 25 *sq. ft.*

20. Required the side of the equilateral triangle inscribed in a circle whose diameter is 4 ft. *Ans.* 3·464 *ft.*

21. Find by trigonometry the perpendicular let fall from the centre of an octagon, upon one of the sides, the length of the side being 6 ft. *Ans.* 7·24264 *ft.*

22. The diameter of the base of an oblique cone is 15 ft., the greatest slant height 14 ft., and the least slant height 13 ft.; required the solidity of the cone. *Ans.* 659·7 *c. ft.*

23. A piece of tin, having the form of a quadrant of a circle, is rolled up so as to form a conical vessel; required its content when the radius of the quadrant is 10 in. *Ans.* 63·368 *c. in.*

24. What surface of tin will be required to make a conical

vessel, open at the top, whose perpendicular height is 8 in., and the diameter of the top 12 in.? *Ans.* 188·496 *sq. in.*

25. How many superficial feet of board will be required to make a rectangular box, closed at the top, whose length is 4 ft., breadth 3 ft., and depth 2 ft.? *Ans.* 52 *sup. ft.*

26. The side of an equilateral triangle is 15 in.; what will be the side of another equilateral triangle containing $\frac{1}{9}$ the area? *Ans.* 5 *in.*

27. Required the diameter of a ball of cast-iron, weighing 24 lbs. *Ans.* 5·657 *in.*

28. A war ship was found to displace 40000 cubic feet of water; required its tonnage. *Ans.* 1116 *tons.*

29. The length of a cylindrical piece of timber is 5 ft., and the diameter of the base is $1\frac{1}{3}$ ft.; what will be its solidity when hewn into a square prism? *Ans.* $4\frac{4}{9}$ *c. ft.*

30. What is the weight of a hollow sphere of cast-iron, whose exterior diameter is 1 foot, and the thickness of the metal $1\frac{1}{2}$ in.? *Ans.* 137·12 *lbs.*

The five following problems are proposed to be solved by the scale and compasses:—

31. The three sides of a field are 120, 160, and 200 links respectively; it is required to find the length of the perpendicular let fall upon the longest side. *Ans.* 96 *links.*

32. Two sides of a triangle are 6 and 9 chains, and the included angle 30°; it is required to find the perpendicular let fall upon the greater side. *Ans.* 3 *chains.*

33. The side of an equilateral parallelogram is 8, and the angle included by the two adjacent sides is 60°; required the length of the greater diagonal. *Ans.* 13·8.

34. The three sides of the triangle ABC, are 24, 40, and 32 respectively; required the radius of the circle, which shall pass through the points A, B, and C. *Ans.* 20.

35. The diagonal of a rectangle is $16\frac{2}{3}$ ft., and the perpendicular let fall upon it from one of the angles is 8 ft.; required the sides. *Ans.* 10 and $13\frac{1}{3}$ *ft.*

The following problems are given to illustrate the subject of the economy of material:—

36. Required the area of a rectangle 8 ft. long, and 4 ft. broad, and also that of a square having an equal perimeter.

Ans. 32 and 36 *sq. ft.*

37. The walls of a sheep-fold had to be built 6 ft. high, and to enclose a space of 400 sq. ft. Required the cost of the walls, 1st when the fold is built in the form of a rectangle whose length is 25 ft., and 2nd when it is built in the form of a square; allowing 4*s.* per sq. yd. for the walling.

Ans. £10 18*s.* 8*d.* and £10 13*s.* 4*d.*

38. A rectangular cistern having a square base is to be covered with sheet lead; required the depth when the side of the base is 5 ft., and the lead contains 75 sup. ft.

Ans. 2·5 *ft.*

39. A cistern, open at the top, having a square base, is to be covered with 300 sup. ft. of lead; required the content of the vessel, 1st when the side of the base is 10 ft., and 2nd when the side of the base is 8 ft. *Ans.* 500 and 472 *c. ft.*

Here the maximum capacity, with the same material, obtains when the height is half the side of the base.

40. A box with its cover is to be made out of 54 sup. in. of material. It is required to find its content, 1st when it has the form of a cube, and 2nd when the sides of the base are 6 and 3 in. respectively. *Ans.* 27 and 18 *c. in.*

41. A piece of ground containing 900 sq. yds. is to be fenced in against a wall already built. Required the length of the new fences, 1st when the length of the ground is twice its breadth, and 2nd when the ground is a square.

Ans. 84·8528 and 90 *yds.*

Here it would be most economical to make the length twice the breadth.

42. The diameter of the base of a cylindrical vessel, *open* at the top, is 10 in. and the height 5 in.; the diameter of *another* cylinder, having the same capacity, is 8 in. Required

the superficial inches in the material composing the two vessels. *Ans.* 235·6 and 246·6.

Here it would be most economical to make the height one half the diameter of the base.

43. The diameter of the base of a cylindrical vessel, ***closed*** at the top, is 10 in., and the height is also 10 in.; the diameter of another cylinder, having the same capacity, is 8 in. Required the surface composing the two vessels. *Ans.* 471·24 and 493·23 *sup. in.*

44. Required the areas of a square and a hexagon, the perimeter of each being 80. *Ans.* 400 and 461·88.

45. The base and perpendicular of a right-angled triangle are 6 and 8 feet respectively; required the difference between its area and that of an equilateral triangle, having an equal perimeter. *Ans.* 3·7128 *sq. ft,*

46. Required the areas of a circle and a square, the perimeter of each being 20. *Ans.* 31·83 and 25.

THE END.

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